Removability of Singular Sets of Harmonic Maps

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Abstract. We prove that a harmonic map with small energy and monotonicity property is smooth if its singular set is rectifiable and has a finite uniform density; moreover, the monotonicity property holds if the singular set has a lower dimension or its gradient has higher integrability. This work generalizes the results in [CL][DF][LG12], which were proved under the assumptions that the singular sets are isolated points or smooth submanifolds.

§ 1. Introduction.

Suppose that $m,\ n \geq 2$ are integers and $1 < p < \infty$. Let $\Omega \subset \mathbf{R}^m$ be a bounded smooth domain and $N \subset \mathbf{R}^n$ be a smooth compact submanifold. Denote by $W^{1,p}(\Omega, N)$ the set of all functions $u \in L^p(\Omega, \mathbf{R}^n)$ with image in N and finite (p-)energy:

$$\text{(1.1)} \qquad \int_\Omega |\nabla u|^p dx < \infty, \quad \text{where } |\nabla u|^2 = \sum_{\alpha,i} \left(\frac{\partial u^i}{\partial x_\alpha}\right)^2 .$$

A (*weakly*) (p-)*harmonic map* from Ω to N is a critical point of (1.1) in $W^{1,p}(\Omega, N)$. A *stationary* (p-)harmonic map [SR2] is a harmonic map that is also a critical point with respect to the deformations of the domain Ω. A map with least energy among those maps in $W^{1,p}(\Omega, N)$ of same boundary data is called (p-)*energy minimizer*. (The prefix p- is added for emphasis.)

It is well-known that a harmonic map, or even a minimizer, may have only *partial regularity*, that is, being regular on the complement of a subset, called *singular set*. For partially regular harmonic maps, it is desirable to know whether they are entirely regular; that is, their singular sets are actually removable.

Sacks and Uhlenbeck [SaU] showed that a 2-harmonic map on $\mathbf{B}^2 \setminus \{0\}$ is smooth on $\mathbf{B}^2$; this holds for m-harmonic maps on $\mathbf{B}^m \setminus \{0\}$ for any $m \geq 2$, as shown in [MY], where $\mathbf{B}^m = \{x \in \mathbf{R}^m : |x| < 1\}$. For p-harmonic maps with small energy, $1 < p < m$, isolated singularities are also removable; this was proved by Liao [LG1] for $p = 2$ and by Duzaar and Fuchs [DF] for $p \geq 2$. For non-isolated case, Costa and Liao proved in [CL] [LG2] that the $m-3$ dimensional singular submanifold of a 2-harmonic map with small energy and monotonicity property is removable.

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Here will study the removability of singular sets with rectifiable structure. We show that a p-harmonic map with small energy and monotonicity property is smooth if its rectifiable apparent singular set has a bounded *uniform density* [Theorem 2.1]. In particular, a singular set that is the union of finite smooth submanifolds of codimension $[p]+1$ and a lower dimensional rectifiable set is removable.

According to the work of Schoen and Uhlenbeck [SU], Hardt and Lin [HL1] and Luckhaus [LS], the singular set of a p-minimizer has Hausdorff dimension $m-[p]-1$. The structure of singular sets could be wild unless $m \leq [p]+1$, in which case they are isolated. Hardt and Lin [HL2] proved that the singular set of a 2-minimizer from $\mathbf{B}^4$ to $\mathbf{S}^2$ is the union of finitely many $C^{0,\alpha}$ curves together with a discrete set. Simon obtained the $C^{1,\alpha}$ regularity of those curves and established the rectifiability of singular sets of 2-minimizers under more general setting; see [SL1,2]. These results partially motivate this paper.

The assumption that the map has monotonicity property is essential to Theorem 2.1. Energy minimizers and stationary harmonic maps have monotonicity property. One the other hand, a weakly harmonic map, for example the one from $\mathbf{B}^3$ to $\mathbf{S}^2$ with a line singular set constructed by Riviere [RT1], has no monotonicity property, for otherwise Evans' work [EL] would implies that the singular set has $\mathcal{H}^1$ measure 0. Nonetheless, we prove that a p-harmonic map has monotonicity property if its singular set has a lower dimension, or its gradient has higher integrability [Theorem 2.2]. Costa and Liao [CL] showed the same result for 2-harmonic maps with smooth singular manifolds. The proof of Theorem 2.2 (c) also shows a monotonicity property of the normalized energy on the tubular neighborhoods of the singular set; see (4.19).

Note that the removable singularity theorems of different forms were proved in [SJ][HP][EP][M] and others. They assert that classical solutions of equations (or systems) on the complement of a small set Z (in certain sense) can be extended across Z to get a weak solution. For single equations (with proper growth conditions), those theorems are complete, as any of their weak solutions are smooth; see [DG] [MC]. For systems, this is not true. A simple example is the map $x \to \frac{x}{|x|}$ from $\mathbf{B}^m$ to $\mathbf{S}^{m-1}$, which is discontinuous at 0 but it is a minimizer for integer $p \in (1, m)$ and in particular it satisfies the system $\Delta u + |\nabla u|^2 u = 0$; see [CG][LF][BCL]. The theorems in this paper fill this gap between partial regularity and everywhere regularity.

Also note that Helélin [HF1,2] proved everywhere regularity of harmonic maps on a 2-surface. The singular sets of p-harmonic maps with monotonicity from $\mathbf{B}^m$ to spheres have $\mathcal{H}^{m-p}$ measure 0; in particular, an m-harmonic map to a sphere is smooth; see [EL] [MY] and also [SR2].

Section 2 contains the precise statements of Theorems 2.1, 2.3, some necessary definitions and notations. Section 3 is devoted to the proof of Theorem 2.1. The key step is to prove the strong convergence of the blow-up sequence by analyzing the asymptotic behaviors near the singular sets. The proof of Theorem 2.2 is given in Section 4.

Also included in this paper (Section 5) is an example of system of equations whose solution has prescribed singular submanifold. This system is uniformly elliptic, quasilinear with quadratic growth and is homogeneous (in the sense that 0 is a solution). This gives a positive partial answer to the questions posed in [G][SR1] on prescribing singular sets.

We remark that the results in this paper hold for the critical points of more general functionals, such as those considered in [GG1,2][FM].

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§ 2. Statement of the Main Results

Definitions and Notations. From the definition, a (weakly) p-harmonic map is also a weak solution in $W^{1,p}(\Omega, N)$ of the Euler-Lagrange equation of (1.1):

$$\text{div}(|\nabla u|^{p-2}\nabla u) - |\nabla u|^{p-2}A(u)(\nabla u, \nabla u) = 0, \tag{2.1}$$

where $A(u)$ is the second fundamental form of N evaluated at u [SU][HL1][DF]. A p-minimizer is a stationary p-harmonic map, and hence is a p-harmonic map.

For a subset $Z \subset \mathbf{R}^m$ and $x \in \mathbf{R}^m$, denote $\rho(x) \equiv \rho(x, Z)$ the distance from x to Z, and for $r > 0$, denote

$$Z_r = \{x \in \mathbf{R}^m : \rho(x, Z) < r\};$$
$$Z^r = \{x \in Z : \rho(x, Z) > r\};$$
$$\mathbf{B}(x, r) = \{y \in \mathbf{R}^m : |y - x| < r\}.$$

We denote by $\mathcal{L}^m$ the Lebesgue measure in $\mathbf{R}^m$ and $\alpha(m) = \mathcal{L}^m[\{x \in \mathbf{R}^m : |x| < 1\}]$.

We usually omit the differential Lebesgue measure dx from our integrals. The constants C_0, C_1, ... depend only on m, n, p, Ω, N, and K in Theorem 2.1; those depending only on m, n and p are called *absolute constants*.

A map $u \in W^{1,p}(\Omega, N)$ is said to have *monotonicity property*, if for each $x \in \Omega$ and $0 < r < \rho(x, \partial\Omega)$, the *normalized energy*

$$E(x, r) = r^{p-m} \int_{\mathbf{B}(x,r)} |\nabla u|^p \tag{2.2}$$

is *increasing* in r, that is,

$$E(x, r) \leq E(x, s) \tag{2.3}$$

for all $0 < r < s < \rho(x, \partial\Omega)$.

Let $q \geq 0$ be an integer. The $m-q$ dimensional *Minkowski content* and *Hausdorff measure* [F1] of a subset $Z \subset \mathbf{R}^m$ are defined, respectively, by

$$\mathcal{M}^{m-q}(Z) = \lim_{r \to 0+} \mathcal{L}^m[Z_r]/[\alpha(q) r^q], \tag{2.4}$$

$$\mathcal{H}^{m-q}(Z) = \alpha(m-q) \inf_{\varepsilon \to 0+} \{ \sum_{r_i < \varepsilon} r_i^{m-q} : Z \subset \cup_i \mathbf{B}(x_i, r_i) \},$$

whenever the limits exist.

A subset $Z \subset \mathbf{R}^m$ is $m-q$ *rectifiable* (or, *rectifiable of codimension* q) [F1, 3.2.14] if and only if it is a Lipschitzian image of a bounded subset of $\mathbf{R}^{m-q}$ onto Z. It follows that $\mathcal{H}^{m-q}(Z) < \infty$ if Z is $m-q$ rectifiable.

We will use the following relation between $\mathcal{H}$ and $\mathcal{M}$.

Theorem 2.0 ([F1, 3.2.39]). *If Z is closed and $m-q$ rectifiable, then $\mathcal{M}^{m-q}(Z) = \mathcal{H}^{m-q}(Z)$.*

We define the $m-q$ *uniform density* $\Psi^{m-q}(Z)$ *of* Z as follows

$$\Psi^{m-q}(Z) = \sup\{ \frac{\mathcal{L}^m[\bar{\mathbf{B}}(x,t) \cap Z_r]}{\alpha(m-q)\alpha(q) t^{m-q} r^q} : x \in \mathbf{R}^m, t, r > 0 \}. \tag{2.5}$$

It is direct to verify that at any $x \in \mathbf{R}^m$,

$$\Phi^{*m-q}(\mathcal{H}^{m-q}, Z, x) \leq \Psi^{m-q}(Z),$$

where $\Phi^{*m-q}(\mathcal{H}^{m-q}, Z, x)$ is the upper *density* at x of Z with respect to the measure $\mathcal{H}^{m-q}$; see [SL1] [F1]. Let r, s, t be any numbers satisfying $0 < r < s < t$. Then by Theorem 2.0 and (2.5),

$$\begin{aligned}
&\mathcal{H}^{m-q}[\bar{\mathbf{B}}(x, t-s) \cap Z] = \mathcal{M}^{m-q}[\bar{\mathbf{B}}(x, t-s) \cap Z] \\
&= \lim_{r \to 0} \frac{\mathcal{L}^m[(\bar{\mathbf{B}}(x, t-s) \cap Z)_r]}{\alpha(q) r^q} \leq \lim_{r \to 0} \frac{\mathcal{L}^m[\bar{\mathbf{B}}(x,t) \cap Z_r]}{\alpha(q) r^q} \\
&\leq \Psi^{m-q}(Z) \alpha(m-q) t^{m-q}.
\end{aligned}$$

Sending $s \to 0$, we get $\mathcal{H}^{m-q}[\bar{\mathbf{B}}(x,t) \cap Z] \leq \Psi^{m-q}(Z)\alpha(m-q)t^{m-q}$, and it follows that $\Phi^{*m-q}(\mathcal{H}^{m-q}, Z, x) \leq \Psi^{m-q}(Z)$.

On the other hand, that Z is $m-q$ rectifiable implies $\Phi^{*m-q}(\mathcal{H}^{m-q}, Z, x) = 1$ for $\mathcal{H}^{m-q}$-a.e. $x \in Z$; see [SL1, 3.6][F1, 3.2.19]. $\Psi(Z)$ can be considered as an upper bound of the density for all $x \in Z$, which is kept under rescaling; see (3.16) and Lemma 3.3.

Our main results are

Theorem 2.1 (Removability of Singular Sets). *Let m, $n \geq 2$ be positive integers, $1 < p < m$. Suppose that $N \subset \mathbf{R}^n$ is a smooth compact submanifold, $\Omega \subset \mathbf{R}^m$ is a bounded domain and $K \subset \Omega$ is a compact subset. Then there is a positive number ε depending only on m, n, p, K and N such that if $u \in W^{1,p}(\Omega, N)$ is a p-harmonic map satisfying the following conditions*

(a). *u has monotonicity property and $\int_\Omega |\nabla u|^p \leq \varepsilon$;*

(b). *$u \in C^1(\Omega \setminus Z, N)$, where $Z \subset \Omega$ is relatively closed with $\Psi^{m-[p]-1}(Z) < \infty$.*

Then $u \in C^1(K, N)$.

Theorem 2.2. *A p-harmonic map $u \in C^1(\Omega \setminus Z, N) \cap W^{1,p}(\Omega, N)$ has monotonicity property, if one of the following holds:*

(a). *Z is closed and rectifiable of Hausdorff codimension $\geq [p] + 1$ and*

$$\int_\Omega |\nabla u|^{p+\frac{p}{[p]}} < \infty. \tag{2.6}$$

(b). *Z is closed and rectifiable of Hausdorff codimension $\geq [p]+2$ and $|\nabla u(x)| \leq C_1/\rho(x, Z)$ for some $C_1 > 0$ and for all $x \in \Omega \setminus Z$.*

(c). *Z is a compact smooth submanifold (say C^2) of codimension $\geq [p] + 2$.*

Remark 2.3.. If $\Omega = \mathbf{B}(0,1)$, $Z = \{0\}$ and $u \in C^1(\mathbf{B}(0,1) \setminus \{0\}, N)$, then $E(0,r)$ is increasing in $r \in (0,1)$ (see [DF][LG1][SaU][MY]). This is sufficient to remove the possible singularity 0, if the energy is small. So when Z is isolated, then the only condition needed in Theorem 2.1 is $\int_\Omega |\nabla u|^p \leq \varepsilon$. In particular, any isolated singularities of m-harmonic maps are removable. See [SaU][DF][MY]. When $\Omega = \mathbf{B}(0,2)$, $p = 2$, $K = \bar{\mathbf{B}}(0,1)$ and Z is a smooth submanifold of $\mathbf{B}(0,2)$, Liao in [LG2] proved the same conclusion of Theorem 1.2.

Remark 2.4. When Z is a smooth submanifold, Costa and Liao [CL] showed Theorem 2.2 (a) (c) for 2-harmonic maps.

§ 3. Proof of the Theorems 2.1

Let $\delta = \operatorname{dist}(K, \partial\Omega)$, then $K \subset \Omega^\delta$. By a standard iteration argument, the proof of Theorem 2.1 is reduced to the following

Lemma 3.1. *There exist numbers $0 < \varepsilon_0, \tau < 1$, depending only on p, m, n, δ and N such that for u satisfying the hypotheses of Theorem 2.1, $x \in \Omega^\delta$ and $0 < r \leq \delta$,*

$$E(x,r) \leq \varepsilon_0 \quad \text{implies} \quad E(x,\tau r) \leq \frac{1}{2} E(x,r).$$

Proof of Theorem 2.1 from Lemma 3.1. Let $\delta = \text{dist}(K, \partial\Omega)$ and choose $\varepsilon = \delta^{m-p}\varepsilon_0$. Suppose that u is as in Theorem 2.1. Note that the monotonicity property and $\int_\Omega |\nabla u|^p \le \varepsilon$ imply

$$E(x,r) = r^{p-n}\int_{\mathbf{B}(x,r)} |\nabla u|^p \le E(x,\delta) \le \delta^{p-m}\int_\Omega |\nabla u|^p \le \varepsilon_0$$

for all $x \in \Omega^\delta$ and $0 < r \le \delta$. By Lemma 3.1, there is a $\tau \in (0,1)$ such that

$$E(x,\tau r) \le \frac{1}{2}E(x,r). \tag{3.1}$$

Let $\theta = \log_\tau \frac{1}{2}$ and $k \ge 1$ be the integer such that $r \in [\tau^k\delta, \tau^{k-1}\delta)$, then by iterating (3.1), we get

$$E(x,r) \le E(x,\tau^{k-1}\delta) \le 2^{-k+1}E(x,\delta) \le 2\varepsilon_0 (r/\delta)^\theta.$$

By Morrey's Lemma [MC, 3.5.2], u is $C^{\theta/p}$ on Ω^δ. That $u \in C^{1,\theta/p}(\Omega^\delta, N)$ follows from the standard argument, see for example [HL1, §3]. ■

Proof of Lemma 3.1. We will use the blow-up argument, as employed in [HKL][HL1][EL]. If the conclusion was not true, then for any $0 < \tau < 1$, there would exist sequences $x_i \in \Omega$ and $0 < r_i \le \delta \le \rho(x_i, \partial\Omega)$ such that

$$\lambda_i^p \equiv E(x_i, r_i) \downarrow 0, \quad \text{but} \quad E(x_i, \tau r_i) \ge \frac{1}{2}\lambda_i^p, \quad i = 1, 2, 3.... \tag{3.2}$$

Denote $a_i = ⨍_{\mathbf{B}(x_i,r_i)} u(x)dx$. Define v_i by

$$v_i(z) = \lambda_i^{-1}[u(x_i + r_i z) - a_i], \quad z \in \mathbf{B}(0,1). \tag{3.3}$$

By the change of variables $z \to x_i + r_i z$ and Poincare inequality, (3.2) and (3.3) imply that

$$\int_{\mathbf{B}(0,1)} |\nabla v_i|^p dz = 1, \quad \int_{\mathbf{B}(0,1)} |v_i|^p dz \le C_2 \int_{\mathbf{B}(0,1)} |\nabla v_i|^p dz \le C_2, \tag{3.4}$$

for an absolute constant C_2, but

$$\tau^{p-m}\int_{\mathbf{B}(0,\tau)} |\nabla v_i|^p dz \ge \frac{1}{2}. \tag{3.5}$$

As a p-harmonic map, u satisfies (2.1) in the sense

$$\int_\Omega |\nabla u|^{p-2}\nabla u \cdot \nabla\varphi + |\nabla u|^{p-2}A(u)(\nabla u, \nabla u)\cdot \varphi dx = 0 \tag{3.6}$$

for each $\varphi \in C_0^1(\Omega, \mathbf{R}^n)$. By the change of variables $z \to x_i + r_i z$, v_i satisfies the rescaled form of (3.6)

$$\int_{\mathbf{B}(0,1)} |\nabla v_i|^{p-2}\nabla v_i \cdot \nabla\varphi dz = -\lambda_i \int_{\mathbf{B}(0,1)} |\nabla v_i|^{p-2}A(a_i + \lambda_i v_i)(\nabla v_i, \nabla v_i)\cdot\varphi dz \tag{3.7$_i$}$$

for all $\varphi \in C_0^1(\mathbf{B}(0,1), \mathbf{R}^n)$. In fact, $(3.7)_i$ holds for all $\varphi \in W_0^{1,p}(\mathbf{B}(0,1), \mathbf{R}^n) \cap L^\infty$, since such functions φ can be approximated by C_0^1 functions (in $W^{1,p}$ norm).

Claim 3.2. *There is a subsequence* $\{v_k\} \subseteq \{v_i\}$ *and a function* $v_0 \in W^{1,p}(\mathbf{B}(0,1), \mathbf{R}^n)$ *such that*

$$v_k \to v_0 \quad \text{in} \quad W^{1,p}(\mathbf{B}(0,1/2), \mathbf{R}^n) \quad \text{(strongly).} \tag{3.8}$$

Completion of Proof from Claim 3.2. Now a contradiction follows from Claim 3.2.

Note that (3.8) and (3.4) imply the following

$$\int_{\mathbf{B}(0,1/2)} |\nabla v_0|^p dz \leq 1, \quad \int_{\mathbf{B}(0,1/2)} |v_0|^p dz \leq C_2, \tag{3.9}$$

$$|\nabla v_k|^{p-2}\nabla v_k \to |\nabla v_0|^{p-2}\nabla v_0 \text{ in } L^{p/p-1}(\mathbf{B}(0,1/2), \mathbf{R}^n),$$

while (3.8) and (3.5) imply

$$\tau^{p-m} \int_{\mathbf{B}(0,\tau)} |\nabla v_0|^p dz \geq \frac{1}{2}. \tag{3.10}$$

Since N is smooth and $\lambda_k \to 0$, there is a constant C_3 depending only N such that

$$\begin{aligned} &|\lambda_k \int_{\mathbf{B}(0,1)} |\nabla v_k|^{p-2} A(a_k + \lambda_k v_k)(\nabla v_k, \nabla v_k) \cdot \varphi dz| \\ &\leq C_3 \lambda_k \sup|\varphi| \int_{\mathbf{B}(0,1/2)} |\nabla v_k|^p dz \leq C_3 \lambda_k \sup|\varphi| \to 0. \end{aligned} \tag{3.11}$$

Using (3.11) and (3.9), we take limit in $(3.7)_k$ to get

$$\int_{\mathbf{B}(0,1/2)} |\nabla v_0|^{p-2}\nabla v_0 \cdot \nabla\varphi dz = 0$$

for all $\varphi \in C_0^1(\mathbf{B}(0,1/2), \mathbf{R}^n)$. So v_0 is a p-harmonic function $W^{1,p}(\mathbf{B}(0,1/2), \mathbf{R}^n)$.

By Theorem 3.2 in [UK] and Theorem 5.1 in [TP], there is an absolute constant C_4 such that

$$\sup_{\mathbf{B}(0,1/4)} |\nabla v_0| \leq C_4 \int_{\mathbf{B}(0,1/2)} |\nabla v_0|^p dz \leq C_4,$$

where (3.9) is used. For $0 < \tau < 1/4$, it follows from this estimate that

$$\tau^{p-m} \int_{\mathbf{B}(0,\tau)} |\nabla v_0|^p dz \leq C_4^p \alpha(m) \tau^p. \tag{3.12}$$

Let us start with a τ less than $\min\{\frac{1}{4}, \frac{1}{(2\alpha(m))^{1/p} C_4}\}$, then (3.12) is a contradiction to (3.10). So Claim (3.2) implies Lemma 3.1. ■

Now we prepare the proof of Claim 3.2. First we take a subsequence $\{j\} \subseteq \{i\}$ such that for some $v_0 \in W^{1,p}(\mathbf{B}(0,1), \mathbf{R}^n)$ it holds that

$$(3.13) \qquad v_j \to v_0 \ \text{ in } L^p(\mathbf{B}(0,1), \mathbf{R}^n); \quad \nabla v_j \rightharpoonup \nabla v_0 \quad \text{weakly in } L^p(\mathbf{B}(0,1), \mathbf{R}^n).$$

We may assume $x_j \to x_0$ for some $x_0 \in \Omega^\delta$.

Consider the subsets

$$(3.14) \qquad Z^j \equiv r_j^{-1}[\bar{\mathbf{B}}(x_j, r_j) \cap Z - x_j] = \{z \in \bar{\mathbf{B}}(0,1) : x_j + r_j z \in Z\}.$$

$Z^j \neq \emptyset$ may be assumed by adding $\{0\}$ to it if necessary. We show that $\mathcal{M}^{m-q}(Z^j) \leq C_0$, where $C_0 = \Psi^{m-q}(Z)\alpha(m-q)$. Note that $\rho(x_j + r_j z, Z) = r_j \rho(z, Z^j)$ for $z \in \mathbf{B}(0,1)$ and

$$(3.15) \qquad Z_r^j \equiv \{x \in \mathbf{R}^m : \rho(x, Z^j) \leq r\} = r_j^{-1}[\bar{\mathbf{B}}(x_j, r_j) \cap Z_{rr_j} - x_j].$$

By (3.15) and (2.5),

$$(3.16) \qquad \mathcal{L}^m[Z_r^j] = r_j^{-m}\mathcal{L}^m[\bar{\mathbf{B}}(x_j, r_j) \cap Z_{rr_j}] \leq C_0\alpha(q) r_j^{-m} r_j^{m-q}(rr_j)^q \leq C_0\alpha(q) r^q.$$

This implies, by definition (2.4), $\mathcal{M}^{m-q}(Z^j) \leq C_0$.

We now have three lemmas.

Lemma 3.3. *There is a subsequence $\{Z^k\} \subseteq \{Z^j\}$ and a compact subset $Z^0 \subset \bar{\mathbf{B}}(0,1)$ such that $d_H(Z^k, Z^0) \to 0$ and $\mathcal{M}^{m-q}(Z^0) \leq C_0$, where d_H is the Hausdorff distance.*

Proof : By the compactness in d_H of a family of compact subsets in the unit ball $\bar{\mathbf{B}}(0,1)$ [F1, 2.10.21], there is a subsequence $\{Z^k\}$ and a compact subset Z^0 such that $d_H(Z^k, Z^0) \to 0$ as $k \to \infty$.

To show that $\mathcal{M}^{m-q}(Z^0) \leq C_0$, let $s > 0$ be any number and k be so large such that $Z^0 \subseteq Z_s^k$. Then $Z_r^0 \subseteq Z_{r+s}^k$ for any $r > 0$, and from (3.16), we have

$$(3.17) \qquad \mathcal{L}^m[Z_r^0] \leq \mathcal{L}^m[Z_{r+s}^k] \leq C_0\alpha(q)(r+s)^q.$$

Since s is arbitrary, $\mathcal{L}^m[Z_r^0] \leq C_0\alpha(q) r^q$. By definition (2.4), $\mathcal{M}^{m-q}(Z^0) \leq C_0$. ∎

Lemma 3.4. *There are constants ε_0, C_5 and C_6 depending only on m, p, δ and N such that if u is as in Theorem 2.1 with $\int_\Omega |\nabla u|^p \leq \varepsilon_0$, then*

$$(3.18) \qquad |\nabla u(x)| \leq C_5 r^{-1} E(x,r)^{1/p};$$

$$(3.19) \qquad |\nabla u(x)| \leq C_6/\rho(x),$$

for any $x \in \Omega^\delta \setminus Z$ and $0 < r \leq \min\{\rho(x), \delta\}$, where $\rho(x) = \rho(x, Z)$.

Proof : From the Theorem 2.1 in [DF], there are numbers ε_1 and C_5, depending only on m and N, such that (3.18) holds as long as $E(x,r) \le \varepsilon_1$. The condition $E(x,r) \le \varepsilon_1$ is now verified by choosing ε_0 properly. Take $\varepsilon_0 = \delta^{m-q}\varepsilon_1$. Then from $\int_\Omega |\nabla u|^p \le \varepsilon_0$ and monotonicity property, we have

$$E(x,r) \le E(x,\delta) \le \delta^{p-m} \int_\Omega |\nabla u|^p \le \varepsilon_1. \tag{3.20}$$

(3.19) is obtained by taking $r = \min\{\rho(x), \delta\}$ in (3.18) and (3.20). When $\rho(x) > \delta$, we need to use the monotonicity property and replace C_5 by a larger number C_6. ■

Lemma 3.5. *There exists a constant $C_7 > 0$ depending on m, n, p, δ, N but independent of k such that if k is large enough, then for all $z \in \mathbf{B}(0,3/4) \setminus Z^k$,*

$$|\nabla v_k(z)| \le C_7/\rho(z, Z^k). \tag{3.21}$$

Furthermore if $z \in \mathbf{B}(0,3/4) \setminus Z^0$, then

$$|\nabla v_0(x)| \le C_7/\rho(z, Z^0). \tag{3.22}$$

Proof : From the definition (3.3) of v_k, we have that for $z \in \mathbf{B}(0,3/4) \setminus Z^k$

$$\nabla v_k(z) = \nabla u(x_k + r_k z) r_k \lambda_k^{-1}.$$

Applying Lemma 3.4 and the monotonicity with $r = \rho(x_k + r_k z, Z)/3 = r_k \rho(z, Z^k)/3 \le r_k/4$ and using (3.2), we obtain

$$\begin{aligned} |\nabla v_k(z)| &\le C_5 r_k \lambda_k^{-1} \rho^{-1}(x_k + r_k z, Z) E(x_k + r_k z, \rho(x_k + r_k z, Z)/3)^{1/p} \\ &\le C_5 \rho^{-1}(z, Z^k) \lambda_k^{-1} E(x_k + r_k z, r_k/4)^{1/p} \\ &\le C_5 \rho^{-1}(z, Z^k) \lambda_k^{-1} [4^{m-p} E(x_k, r_k)]^{1/p} \\ &= C_5 4^{m/p-1} \rho^{-1}(z, Z^k) = C_7/\rho(z, Z^k). \end{aligned} \tag{3.23}$$

To show (3.22), let $s > r > 0$ be any numbers and let k be so large that $Z^k \subseteq Z_r^0$, then for $z \in \mathbf{B}(0,3/4) \setminus Z_s^0$,

$$\rho(z, Z^k) \ge \rho(z, Z_r^0) \ge \rho(z, Z^0) - r \ge s - r. \tag{3.24}$$

By (3.23), for $z \in \mathbf{B}(0,3/4) \setminus Z_s^0$,

$$|\nabla v_k(z)| \le C_7/\rho(z, Z^k) \le C_7/[\rho(z, Z^0) - r] \le C_7/[s-r]. \tag{3.25}$$

This implies that $v_k(z) \to v_0(z)$ uniformly in $\mathbf{B}(0,3/4) \setminus Z_s^0$ (at least for a subsequence). Thus (3.25) in turns implies for $z \in \mathbf{B}(0,3/4) \setminus Z_s^0$,

$$|\nabla v_0(z)| \le C_7/[\rho(z, Z^0) - r].$$

Since $s > r > 0$ are arbitrary, (3.22) follows. ■

Lemma 3.6 [HM, 2.1]. *If* $\Sigma \subset \mathbf{R}^m$ *with* $\mathcal{M}^{m-q}(\Sigma) < \infty$, $q > \nu$, *then*

$$\int_{\Sigma_r} \rho(z,\Sigma)^{-\nu} dz \le C_8 r^{q-\nu}, \quad C_8 = \alpha(q)\mathcal{M}^{m-q}(\Sigma)2^\nu/(1-2^{\nu-q}). \tag{3.26}$$

Proof : By the definition (2.4),

$$\begin{aligned}\int_{\Sigma_r} \rho(z,\Sigma)^{-\nu} dz &\le \sum_{i=0}^{\infty} \int_{\Sigma_{2^{-i}r}\setminus\Sigma_{2^{-i-1}r}} \rho(z,\Sigma)^{-\nu} dz \\ &\le \sum_{i=0}^{\infty} (2^{-i-1}r)^{-\nu}\mathcal{L}^m[\Sigma_{2^{-i}r}] \\ &\le \alpha(q)\mathcal{M}^{m-q}(\Sigma)\sum_{i=0}^{\infty}(2^{-i-1}r)^{-\nu}(2^{-i}r)^q \\ &= \alpha(q)\mathcal{M}^{m-q}(\Sigma)2^\nu/(1-2^{\nu-q})r^{q-\nu}.\end{aligned}$$

∎

Proof of Claim (3.2). Now we show that $\nabla v_k \to \nabla v_0$ strongly in $W^{1,p}(\mathbf{B}(0,1/2),\mathbf{R}^n)$.

For any $r > 0$, by lemma 3.3, there is a number $K(r)$ such that $Z^0 \subseteq Z_r^k$ when $k \ge K(r)$. It follows that $Z_r^0 \subseteq Z_{2r}^k$. By (3.16) and (3.21) and (3.26) with $q = [p]+1$,

$$\int_{Z_r^0} |\nabla v_k|^p \le \int_{Z_{2r}^k} |\nabla v_k|^p \le C_7^p \int_{Z_{2r}^k} \rho(z,Z^k)^{-p} \le C_8 r^{[p]+1-p}. \tag{3.27}$$

Therefore for any given $\mu > 0$, we can choose $s > 0$ so small that for all $k \ge K(s)$,

$$\int_{Z_s^0} |\nabla v_k|^p < \mu. \tag{3.28}$$

Let $\eta(z) = \eta_1(\rho(z))\eta_2(|z|)$, where $\rho(z) = \rho(z,Z^0)$, and $\eta_1(\rho) : [0,\infty) \to [0,1]$ and $\eta_2(r) : [0,1] \to [0,1]$ are cutoff functions satisfying

$$\eta_1(\rho) = 0 \text{ for } 0 \le \rho \le s/2; \quad \eta_1(\rho) = 1 \text{ for } \rho \ge s, \text{ and } |\nabla\eta_1| \le 3s^{-1},$$

$$\eta_2(r) = 1 \text{ for } 0 \le r \le 1/2; \quad \eta_2(r) = 0 \text{ for } r \ge 3/4, \text{ and } |\nabla\eta_2| \le 3.$$

Recall that for $n \ge 1$ and $1 < p < \infty$, there is a number $c = c(n,p) > 0$ such that if $a, b \in \mathbf{R}^n$, then

$$\big(|a|^{p-2}a - |b|^{p-2}b\big)\cdot(a-b) \ge c|a-b|^p, \quad \text{if } p \ge 2; \tag{3.29}$$

$$(3.30)\qquad \left(|a|^{p-2}a-|b|^{p-2}b\right)\cdot(a-b)\geq c|a-b|^2(|a|+|b|)^{p-2},\quad \text{if } 1<p<2.$$

For a proof, see for example [AF]. When $p\geq 2$, by (3.29) and the equations $(3.7)_l$ and $(3.7)_k$, we have,

$$\begin{aligned}
&c\int_{\mathbf{B}(0,1/2)\setminus Z_s^0}|\nabla v_l-\nabla v_k|^p\leq\int_{\mathbf{B}(0,1)}(|\nabla v_l|^{p-2}\nabla v_l-|\nabla v_k|^{p-2}\nabla v_k)\cdot\eta\nabla(v_l-v_k)\\
&\leq\int_{\mathbf{B}(0,1)}(|\nabla v_l|^{p-2}\nabla v_l-|\nabla v_k|^{p-2}\nabla v_k)\cdot\nabla[\eta(v_l-v_k)]\\
&\quad-\int_{\mathbf{B}(0,1)}(|\nabla v_l|^{p-2}\nabla v_l-|\nabla v_k|^{p-2}\nabla v_k)\cdot(v_l-v_k)\nabla\eta\\
&\leq\int_{\mathbf{B}(0,1)}[\lambda_l|\nabla v_l|^{p-2}A(a_l+r_lv_l)(\nabla v_l,\nabla v_l)-\lambda_k|\nabla v_k|^{p-2}A(a_k+r_kv_k)(\nabla v_k,\nabla v_k)]\eta(v_l-v_k)\\
&\quad+\max|\nabla\eta|\int_{\mathbf{B}(0,1)}\left(|\nabla v_l|^{p-1}+|\nabla v_k|^{p-1}\right)|v_l-v_k|\\
&\leq C_9\int_{\mathbf{B}(0,3/4)\setminus Z_{s/2}^0}(\lambda_l|\nabla v_l|^p+\lambda_k|\nabla v_k|^p)\,|v_l-v_k|+C_{10}s^{-1}\int_{\mathbf{B}(0,1)}|v_l-v_k|^p.
\end{aligned}$$

Now using (3.13), (3.25) and the uniform convergence $v_l-v_k\to 0$ on $\mathbf{B}(0,3/4)\setminus Z_{s/2}^0$, we have, when l and k are large,

$$\int_{\mathbf{B}(0,1/2)\setminus Z_s^0}|\nabla v_l-\nabla v_k|^p\leq\mu.$$

This, combined with (3.28), shows that $\int_{\mathbf{B}(0,1/2)}|\nabla v_l-\nabla v_k|^p\leq 2\mu$ when k and l are large. Thus ∇v_k is a Cauchy sequence, and so $\nabla v_k\to\nabla v_0$ in $L^p(\mathbf{B}(0,1/2),\mathbf{R}^n)$.

If $1<p<2$, we use (3.30) to get

$$\begin{aligned}
&c^{\frac{p}{2}}\int_{\mathbf{B}(0,1/2)\setminus Z_s^0}|\nabla v_k-\nabla v_l|^p\\
&\leq c^{\frac{p}{2}}\int_{\mathbf{B}(0,1)}\eta^{\frac{p}{2}}|\nabla v_k-\nabla v_l|^p(|\nabla v_k|+|\nabla v_l|)^{\frac{(p-2)p}{2}}(|\nabla v_k|+|\nabla v_l|)^{\frac{(2-p)p}{2}}\\
&\leq\left(c\int_{\mathbf{B}(0,1)}\eta|\nabla v_k-\nabla v_l|^2(|\nabla v_k|+|\nabla v_l|)^{(p-2)}\right)^{\frac{p}{2}}\left(\int_{\mathbf{B}(0,1)}(|\nabla u|+|\nabla v|)^p\right)^{\frac{2-p}{2}}\\
&\leq\left(\int_{\mathbf{B}(0,1)}[|\nabla v_k|^{p-2}\nabla v_k-|\nabla v_l|^{p-2}\nabla v_l]\cdot[\nabla(v_k-v_l)]\eta\right)^{\frac{p}{2}}\left(\int_{\mathbf{B}(0,1)}(|\nabla u|+|\nabla v|\right)^{\frac{2-p}{2}}.
\end{aligned}$$

The rest of the proof follows as above. ■

§ 4. Proof of Theorem 2.2

It is well-known that a p-harmonic map $C^1(\Omega, N)$ or a stationary p-harmonic map in $W^{1,p}(\Omega, N)$ satisfies

$$\int_\Omega \left(|\nabla u|^p \mathrm{div} X - p|\nabla u|^{p-2} D_\alpha u D_\beta u D_\alpha X^\beta\right) = 0 \tag{4.1}$$

for all $X \in C_0^{0,1}(\Omega, \mathbf{R}^m)$, the space of Lipschitzian functions with compact supports; see [HS][DF][SR2].

From (4.1) one can easily derive monotonicity property, as follows. Suppose $\mathbf{B}(0,\tau) \subset \Omega$. Let $X(x) = \eta(r)x$, where $r = |x|$ and

$$\eta(r) = \begin{cases} 1, & r \le \tau; \\ (h + \tau - r)/h, & \tau \le r \le \tau + h; \\ 0, & r \ge \tau + h. \end{cases}$$

Putting X into (4.1) and taking the limit as $h \to 0+$, one gets

$$(p-m)\int_{\mathbf{B}(0,\tau)} |\nabla u|^p + \tau \int_{\partial \mathbf{B}(0,\tau)} |\nabla u|^p = \tau p \int_{\partial \mathbf{B}(0,\tau)} |\nabla u|^{p-2} |\frac{\partial u}{\partial r}|^2 \ge 0,$$

which implies that

$$\frac{d}{d\tau}\left(\tau^{p-m}\int_{\mathbf{B}(x,\tau)} |\nabla u|^p\right) \ge 0.$$

■

Lemma 4.1. *For a p-harmonic map $u \in C^1(\Omega \setminus Z, N) \cap W^{1,p}$, the identity (4.1) will hold if when $\sigma \to 0+$,*

$$\int_{Z_\sigma} |\nabla u|^p = o(\sigma). \tag{4.2}$$

Proof. Suppose that $X \in C_0^{0,1}(\Omega, \mathbf{R}^m)$. For $0 < \sigma < 1$, let $\xi : [0, \infty) \to [0, 1]$ be a cutoff function defined by

$$\xi(\rho) = \begin{cases} 0, & \rho \le \sigma; \\ (\rho - \sigma)/\sigma, & \sigma \le \rho \le 2\sigma; \\ 1, & \rho \ge 2\sigma. \end{cases}$$

Then $\xi(\rho(x))X(x) \in C_0^{0,1}(\Omega \setminus Z_\sigma, \mathbf{R}^m)$, where $\rho(x) = \rho(x, Z)$. Since $u \in C^1(\Omega \setminus Z_\sigma, N)$, (4.1) holds with X replaced by ξX and yields

$$\int_\Omega \left(|\nabla u|^p \mathrm{div}[\xi X] - p|\nabla u|^{p-2} D_\alpha u D_\beta u D_\alpha [\xi X^\beta]\right) = 0. \tag{4.3}$$

By the definition of ξ, (4.3) implies

$$\int_\Omega \left(\xi|\nabla u|^p \mathrm{div} X - p\xi|\nabla u|^{p-2} D_\alpha u D_\beta u D_\alpha X^\beta\right) \tag{4.4}$$
$$= -\sigma^{-1} \int_{Z_{2\sigma}\setminus Z_\sigma} |\nabla u|^p X \cdot \nabla\rho + p\sigma^{-1} \int_{Z_{2\sigma}\setminus Z_\sigma} |\nabla u|^{p-2} D_\alpha u D_\beta u \rho_\alpha X^\beta,$$

where $\rho_\alpha = D_\alpha \rho$ and $\nabla\rho = (\rho_1, \cdots, \rho_m)$. Taking the limit of the left hand side of (4.4) as $\sigma \to 0$, we get

$$\lim_{\sigma\to 0} (\text{LHS of } (4.4)) = \int_\Omega \left(|\nabla u|^p \mathrm{div} X - p|\nabla u|^{p-2} D_\alpha u D_\beta u D_\alpha X^\beta\right), \tag{4.5}$$

and by using that $|\nabla\rho| \le 1$ and $|D_\alpha u D_\beta u \rho_\alpha X^\beta| \le |\nabla u|^2 |X|$, we have

$$|(\text{RHS of } (4.4))| \le (p+1) \sup|X| \sigma^{-1} \int_{Z_{2\sigma}\setminus Z_\sigma} |\nabla u|^p. \tag{4.6}$$

From (4.4)-(4.6), it is now clear that (4.2) implies (4.1). ■

Thus to prove Theorem 2.2, it suffices to show (4.2).

Proof of Theorem 2.2 (a). Suppose $\int_\Omega |\nabla u|^{p+\frac{p}{[p]}} < \infty$. By Theorem 2.0, $\mathcal{H}^{m-q}(Z) = \mathcal{M}^{m-q}(Z) < \infty$ with $q = [p]+1$; so by (2.4), $\mathcal{L}^m[Z_\sigma] = O(\sigma^{[p]+1})$. Now Hölder's inequality implies that

$$\int_{Z_\sigma} |\nabla u|^p \le \left(\int_{Z_\sigma} |\nabla u|^{p+\frac{p}{[p]}}\right)^{\frac{[p]}{[p]+1}} \mathcal{L}^m[Z_\sigma]^{\frac{1}{[p]+1}} = o(\sigma)$$

as $\sigma \to 0$. This shows (4.2).

■

Proof of Theorem 2.2 (b). Suppose that $|\nabla u(x)| \le C_1/\rho(x)$ for $x \in \Omega \setminus Z$. Since $[p]+2 > p+1$, we can apply (3.26) with $q = [p]+2$ to get

$$\int_{Z_\sigma} |\nabla u|^p \le C_1 \int_{Z_\sigma} \rho^{-p} \le C_2 \sigma^{[p]+2-p} = o(\sigma).$$

■

To verify (4.2) in the case (c) of Theorem 2.2, we need the following lemma, which will also be used in Section 5

Lemma 4.2. *Suppose that $Z \subset \mathbf{R}^m$ is a smooth (say C^2) compact manifold of codimension $q > 0$. There is a number $\delta > 0$ depending only on Z such that for every $x \in Z_\delta$, there is a unique point $\pi(x) \in Z$ such that $\rho(x, Z) = |x - \pi(x)|$, and there is a coordinate system $e_1, \ldots, e_m$ at $\pi(x)$ such that $e_{q+1}, \ldots, e_m$ form an orthonormal base of $T_{\pi(x)}N$, and*

$$\left| \left(\frac{1}{2}\rho^2(x)_{ij} \right)_{m\times m} - \begin{pmatrix} I_{q\times q} & 0 \\ 0 & 0 \end{pmatrix} \right| \leq O(\rho(x)). \tag{4.7}$$

Proof : As noted in [HL1], there are positive numbers δ and C_3 depending only on Z such that for every $x \in Z_\delta$, there is a unique $\pi(x) \in Z_\delta$ such that $\rho(x) = |x - \pi(x)|$ and

$$\|D\pi(x) - P_{\pi(x)}\| \leq C_3|\rho(x)|, \tag{4.8}$$

where $P_{\pi(x)}$ is the orthogonal projection from $\mathbf{R}^m$ to $T_{\pi(x)}Z$. Note that π is at least C^1. Also, for $x \in Z_\delta$,

$$\nabla\rho(x) = \frac{x - \pi(x)}{\rho(x)}. \tag{4.9}$$

For a proof, see [F2, Thm 4.8].

For a fixed $x_0 \in Z_\delta$, we may assume that $\pi(x_0) = 0$. Choose a coordinate system $e_1, \ldots, e_m$ at 0 such that $e_{q+1}, \ldots, e_m$ form an orthonormal base of T_0Z, then (4.8) implies for $y \in \mathbf{R}^m$,

$$|D\pi(x_0)y - P_{\pi(x_0)}y| \leq C_3\rho(x_0)|y|. \tag{4.10}$$

Note that $P_{\pi(x_0)}y = (0, \ldots, 0, y_{q+1}, \ldots, y_m)$. By (4.9) and (4.10)

$$\begin{aligned} \nabla\left(\frac{1}{2}\rho^2(x_0 + y)\right) &= (x_0 + y) - \pi(x_0 + y) = x_0 + y - D\pi(x_0)y + o(|y|) \\ &= x_0 + (y_1, \ldots, y_q, 0, \ldots, 0) + O(\rho(x_0))|y| + o(|y|). \end{aligned} \tag{4.11}$$

Differentiating (4.11) to y and evaluating at $y = 0$, we obtain

$$\left| \left(\frac{1}{2}\rho^2(x_0)_{ij} \right)_{m\times m} - \begin{pmatrix} I_{q\times q} & 0 \\ 0 & 0 \end{pmatrix} \right| \leq O(\rho(x_0)).$$

■

Proof of Theorem 2.2 (c). For $0 < 2\gamma < \sigma < \tau$, let $\eta_\gamma \equiv \eta_{\gamma,\sigma,\tau} : [0,\infty) \to [0,1]$ be a cutoff-function defined by

$$\eta_\gamma(\rho) = \begin{cases} 0, & 0 \le \rho \le \gamma; \\ (\rho-\gamma)/\gamma, & \gamma \le \rho \le 2\gamma; \\ 1, & 2\gamma \le \rho \le \sigma; \\ (\tau-\rho)/(\tau-\sigma), & \sigma \le \rho \le \tau; \\ 0, & \rho \ge \tau. \end{cases}$$

Then (4.1) holds for $X(x) = \eta_\gamma(\rho(x))\nabla(\frac{1}{2}\rho^2(x)) \in C_0^{0,1}(\Omega \setminus Z_\gamma, \mathbf{R}^m)$. Using that $|\nabla\rho| = 1$, we compute

$$\text{div}X = \eta_\gamma'\rho + \eta_\gamma \Delta \frac{1}{2}\rho^2. \tag{4.12}$$

$$D_\alpha X^\beta = \eta_\gamma'\rho\rho_\alpha\rho_\beta + \eta_\gamma \left(\frac{1}{2}\rho^2\right)_{\alpha\beta}. \tag{4.13}$$

Now (4.1) with (4.12) and (4.13) yields

$$\begin{aligned} &\int_\Omega \left(|\nabla u|^p \eta_\gamma'\rho + |\nabla u|^p \eta_\gamma \Delta \frac{1}{2}\rho^2\right) \\ &- p\int_\Omega \left[|\nabla u|^{p-2}\eta_\gamma'\rho|\nabla\rho\cdot\nabla u|^2 + |\nabla u|^{p-2} D_\alpha u D_\beta u \eta_\gamma \left(\frac{1}{2}\rho^2\right)_{\alpha\beta}\right] = 0. \end{aligned} \tag{4.14}$$

Sending $\gamma \to 0$ in (4.14), we get

$$\begin{aligned} &\int_\Omega \left(|\nabla u|^p \eta_0'\rho + |\nabla u|^p \eta_0 \Delta \frac{1}{2}\rho^2\right) \\ &- p\int_\Omega \left[|\nabla u|^{p-2}\eta_0'\rho|\nabla\rho\cdot\nabla u|^2 + |\nabla u|^{p-2} D_\alpha u D_\beta u \eta_0 \left(\frac{1}{2}\rho^2\right)_{\alpha\beta}\right] = 0. \end{aligned} \tag{4.15}$$

By (4.7), for any $\varepsilon > 0$, there is a $\delta(\varepsilon) > 0$ such that for all $x \in Z_\delta$,

$$|\Delta \frac{1}{2}\rho^2(x) - q| \le \varepsilon, \tag{4.16}$$

and

$$D_\alpha u D_\beta u \left(\frac{1}{2}\rho^2(x)\right)_{\alpha\beta} \le (1+\varepsilon)|\nabla u|^2(x). \tag{4.17}$$

Dropping the third term in (4.15) which is nonnegative, and substituting (4.16)–(4.17) into (4.15), we get for $0 < \sigma < \tau \le \delta(\varepsilon)$,

$$-(\tau-\sigma)^{-1}\int_{Z_\tau\setminus Z_\sigma} |\nabla u|^p\rho + [q - \varepsilon - p(1+\varepsilon)]\int_{Z_\tau} \eta_0|\nabla u|^p \le 0. \tag{4.18}$$

Since $q = [p] + 2$, we may write $q - \varepsilon - p(1+\varepsilon) \equiv 1 + p_\varepsilon$, with

$$p_\varepsilon \equiv [p] + 1 - p - (p+1)\varepsilon > 0$$

if ε is chosen so small. Sending $\tau \to \sigma$ in (4.18), we get, for $0 < \sigma \le \delta = \delta(\varepsilon)$,

$$-\sigma \frac{d}{d\sigma} \int_{Z_\sigma} |\nabla u|^p + [1 + p_\varepsilon] \int_{Z_\sigma} |\nabla u|^p \le 0,$$

or, equivalently,

$$\frac{d}{d\sigma}\left(\sigma^{-1-p_\varepsilon} \int_{Z_\sigma} |\nabla u|^p\right) \ge 0. \tag{4.19}$$

(4.19) implies, as $\sigma \to 0$,

$$\sigma^{-1} \int_{Z_\sigma} |\nabla u|^p \le \sigma^{p_\varepsilon} \delta^{-1-p_\varepsilon} \int_{Z_\delta} |\nabla u|^p \to 0.$$

So (4.2) holds. ■

§5. An Example of Elliptic System with Singular Solution

Finally we give an example of elliptic system whose solution is singular on a prescribed submanifold. This shows that, in certain sense, the assumptions of Theorem 2.1 are necessary. Also it gives a positive partial answer to the question posed by Giaquinta [G, p118]: Choose $Z \subset \Omega$ with $\mathcal{H}^{m-3}(Z) < \infty$, does an elliptic system exist with the solution having exactly Z as singular set?

Example. *If $Z \subset \mathbf{R}^m$ is an any smooth (say C^k, $k \ge 3$) compact submanifold of codimension $q \ge 3$, then there is a quasilinear elliptic system of the form*

$$\Delta u^i = \left(\sum_{j=1}^m a_{ij}(x) u^j\right) |\nabla u|^2 + b^i(x) \mathrm{div} u, \quad i = 1, 2, ..., m, \tag{5.1}$$

where $a_{ij} \in C^{k-2}(\Omega, \mathbf{R}^{m^2})$, $b_i \in C^{k-3}(\Omega, \mathbf{R}^m)$ and $\Omega = Z_\tau$ for some $\tau > 0$, which has a weak solution $u \in C^{k-1}(\Omega \setminus Z, \mathbf{S}^{m-1}) \cap H^1(\Omega, \mathbf{S}^{m-1})$ with singular set Z and $\int_\Omega |\nabla u|^2 \le C\tau^{q-2}$ for some $C > 0$.

In fact the gradient of the distance function $\rho(x) = \rho(x, Z)$ solves such equation. As noted in the proof of Lemma 4.2, there is a number $\tau > 0$ so that for $x \in Z_\tau$, there is a unique point $\pi(x) \in Z$ satisfying $\rho(x) \equiv \rho(x, Z) = |x - \pi(x)|$. Also, since $Z \in C^k$, $\rho \in C^k(Z_\tau \setminus Z, \mathbf{R}) \cap C^{0,1}(Z_\tau, \mathbf{R})$ and $\pi \in C^{k-1}(Z_\tau, \mathbf{R}^m)$.

Let $u(x) = \nabla\rho(x)$, then $u \in C^{k-1}(Z_\tau \setminus Z, \mathbf{R}^m)$. Also, u has the following properties [F2, Thm 4.8] : for $x \in Z_\tau \setminus Z$,

$$|u(x)| = 1, \quad u(x) = \nabla\rho(x) = \frac{x - \pi(x)}{\rho(x)}. \tag{5.2}$$

We now show that u satisfies (5.1) with proper choice of a_{ij} and b_i. For simplicity, we again use sub-indices to stand for partial derivatives, and super-indices for vector components; also we employ the summation: repeated indices are summed. Thus from (5.2), for $x \in Z_\tau \setminus Z$, we have

$$u^i = \rho_i = (x^i - \pi^i(x))\rho^{-1}, \quad \text{or} \quad \pi^i(x) = x^i - (\frac{1}{2}\rho(x)^2)_i, \tag{5.3}$$

and

$$\begin{aligned} u^i_\alpha = \frac{\partial u^i}{\partial x^\alpha}(x) =& (\delta^{i\alpha} - \pi^i_\alpha)\rho^{-1} - (x^i - \pi^i)(x^\alpha - \pi^\alpha)\rho^{-3} \\ =& (\delta^{i\alpha} - \pi^i_\alpha - u^i u^\alpha)\rho^{-1}, \end{aligned} \tag{5.4}$$

where δ is the Kronecker index. From (5.4) we get

$$\text{div } u = \sum_i (\delta^{ii} - \pi^i_i - u^i u^i)\rho^{-1} = (m - 1 - \text{div}\pi)\rho^{-1}. \tag{5.5}$$

and

$$\begin{aligned} |\nabla u|^2 =& \sum_{i\alpha} [(\delta^{i\alpha} - \pi^i_\alpha) - u^i u^\alpha]^2 \rho^{-2} \\ &= [(\delta^{i\alpha} - \pi^i_\alpha)^2 + (u^i)^2(u^\alpha)^2 - 2(\delta^{i\alpha} - \pi^i_\alpha)u^i u^\alpha]\rho^{-2} \\ &= [m - 2\text{div}\pi + |\nabla\pi|^2 + 1 - 2]\rho^{-2} \\ &= [m - 1 + 2\text{div}\pi + |\nabla\pi|^2]\rho^{-2}, \end{aligned} \tag{5.6}$$

where we used the following equality from (5.3),

$$\begin{aligned} &2(\delta^{i\alpha} - \pi^i_\alpha)u^i u^\alpha = 2(x^i - \pi^i)_\alpha u^i u^\alpha \\ &= 2(u^i\rho)_\alpha u^i u^\alpha = 2(u^i)^2(u^\alpha)^2 + \rho u^\alpha [\sum_i (u^i)^2]_\alpha = 2. \end{aligned}$$

Using that $\Delta\rho^{-1} = (\text{div}\pi - m + 3)\rho^{-3}$, we obtain from (5.3)

$$\begin{aligned} \Delta u^i &= \Delta(x^i - \pi^i)\rho^{-1} + 2\nabla(x^i - \pi^i) \cdot \nabla(\rho^{-1}) + (x^i - \pi^i)\Delta(\rho^{-1}) \\ &= -\Delta\pi^i\rho^{-1} - 2(u^i - u \cdot \nabla\pi^i)\rho^{-2} + u^i(\text{div}\pi - m + 3)\rho^{-2} \\ &= -\Delta\pi^i\rho^{-1} + 2u \cdot \nabla\pi^i\rho^{-2} + u^i(\text{div}\pi - m + 1)\rho^{-2}. \end{aligned} \tag{5.7}$$

Now (5.5)-(5.7) imply that on $Z_\tau \setminus Z$,

$$\Delta u^i = \sum_j a_{ij}(x) u^j |\nabla u|^2 + b^i(x) \mathrm{div} u, \tag{5.8}$$

where

$$a_{ij} = \frac{2\pi^i_j + (\mathrm{div}\pi - m + 1)\delta^{ij}}{m - 1 - 2\mathrm{div}\pi + |\nabla\pi|^2},$$

$$b_i = -\frac{\Delta\pi^i}{m - 1 - \mathrm{div}\pi}.$$

By Lemma 3.6 and (5.3),

$$\mathrm{div}\pi(x) = m - q + O(\rho(x)), \quad |\nabla\pi(x)|^2 = m - q + O(\rho(x)).$$

Therefore a_{ij} and b_i are well-defined, since their denominators equal to $q - 1 + O(\rho(x))$, which are nonzero for $x \in Z_\tau$ with small $\tau > 0$ and $q \geq 3$. Thus that $\pi \in C^{k-1}$ implies that $a_{ij} \in C^{k-2}(Z_\tau, \mathbf{R}^{m^2})$ and $b_i \in C^{k-3}(Z_\tau, \mathbf{R}^n)$.

Again from (5.3), we have for some constant $C_5 > 0$, $|\nabla u(x)| \leq C_5/\rho(x)$. By (3.26),

$$\int_{Z_\tau} |\nabla u|^2 \leq C_5^2 \int_{Z_\tau} \rho(x)^{-2} \leq C_6 \tau^{q-2}.$$

So $u \in H^1(Z_\tau, \mathbf{R}^m)$.

By removable singularity theory ([M], for example), u is a weak solution on Z_τ. ■

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