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TWO-PERSON ZERO-SUM LINEAR QUADRATIC STOCHASTIC DIFFERENTIAL GAMES BY A HILBERT SPACE METHOD

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Abstract. An open-loop two-person zero-sum linear quadratic (LQ for short) stochastic differential game is considered. The controls for both players are allowed to appear in both the drift and diffusion of the state equation, the weighting matrices in the payoff/cost functional are not assumed to be definite/non-singular, and the cross-terms between two controls are allowed to appear. A forward-backward stochastic differential equation (FBSDE, for short) and a generalized differential Riccati equation are introduced, whose solvability leads to the existence of the open-loop saddle points for the corresponding two-person zero-sum LQ stochastic differential game, under some additional mild conditions. The main idea is a thorough study of general two-person zero-sum LQ games in Hilbert spaces.

1. **Introduction.** Throughout this paper, we let $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ be a complete filtered probability space satisfying the usual condition ([17]), on which a one-dimensional standard Brownian motion $W(\cdot)$ is defined with $\mathbb{F} \equiv \{\mathcal{F}_t\}_{t\geq 0}$ being its natural filtration augmented by all the $\mathbb{P}$-null sets in $\mathcal{F}$. Let $T > 0$ and $x \in \mathbb{R}^n$. Consider the following controlled linear stochastic differential equation (SDE, for short) on time interval $[0, T]$:

$$\begin{cases} dX(t) = [A(t)X(t) + B_1(t)u_1(t) + B_2(t)u_2(t)]dt \\ \qquad\qquad +[C(t)X(t) + D_1(t)u_1(t) + D_2(t)u_2(t)]dW(t), \quad t \in [0, T], \\ X(0) = x. \end{cases} \tag{1.1}$$

In the above $X(\cdot)$, $u_1(\cdot)$, and $u_2(\cdot)$ are $\mathbb{F}$-adapted processes taking values in $\mathbb{R}^n$, $\mathbb{R}^{m_1}$, and $\mathbb{R}^{m_2}$, respectively; and they represent the state and the controls of the two players, respectively. We assume that $A(\cdot)$, $B_1(\cdot)$, $B_2(\cdot)$, $C(\cdot)$, $D_1(\cdot)$, and $D_2(\cdot)$ are deterministic bounded matrix-valued functions of proper dimensions. Clearly, for any $x \in \mathbb{R}^n$ and proper processes $u_1(\cdot)$ and $u_2(\cdot)$ (see the next section), the state equation (1.1) admits a unique strong solution $X(\cdot) \equiv X(\cdot\,; x, u_1(\cdot), u_2(\cdot))$ (which

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is square integrable). Therefore, one can introduce a quadratic performance index (representing the payoff/cost):

$$J_x(u_1(\cdot),u_2(\cdot))=\mathbb{E}\left[\int_s^T q(t,X(t),u_1(t),u_2(t))dt+\langle\, GX(T),X(T)\,\rangle\right]. \tag{1.2}$$

Here,

$$q(t,x,u_1,u_2)=\langle \begin{pmatrix} Q(t) & S_1^\top(t) & S_2^\top(t)\\ S_1(t) & R_{11}(t) & R_{12}(t)\\ S_2(t) & R_{21}(t) & R_{22}(t)\end{pmatrix}\begin{pmatrix} x\\ u_1\\ u_2\end{pmatrix},\begin{pmatrix} x\\ u_1\\ u_2\end{pmatrix}\rangle, \tag{1.3}$$

with $Q(\cdot)$, $S_i(\cdot)$, and $R_{ij}(\cdot)$ being deterministic bounded matrix-valued functions of suitable dimensions and G being a constant matrix.

Roughly speaking, in the game, Player 1 (who takes control $u_1(\cdot)$) wishes to minimize (1.2), and Player 2 (who takes control $u_2(\cdot)$) wishes to maximize (1.2). Therefore, (1.2) represents the cost for Player 1 and the payoff for Player 2. There are basically two types of controls for both players: open-loop controls and closed-loop controls. In this paper, we concentrate on the open-loop case. The closed-loop problem will be addressed in a forthcoming paper.

Let us now look at the case that both players use open-loop controls. Since both players are non-cooperative, they would like to seek their admissible controls $\hat u_1(\cdot)$ and $\hat u_2(\cdot)$, respectively, such that

$$J_x(\hat u_1(\cdot),u_2(\cdot))\le J_x(\hat u_1(\cdot),\hat u_2(\cdot))\le J_x(u_1(\cdot),\hat u_2(\cdot)), \tag{1.4}$$

for all admissible controls $u_1(\cdot)$ and $u_2(\cdot)$ (see the next section for a precise definition). The reason is that when (1.4) holds, none of the players can improve his/her outcome $J_x(\hat u_1(\cdot),\hat u_2(\cdot))$ by deviating from $\hat u_1(\cdot)$ or $\hat u_2(\cdot)$ unilaterally. Thus, both players will be satisfied with the controls $\hat u_1(\cdot)$ and $\hat u_2(\cdot)$, respectively. We refer to $(\hat u_1(\cdot),\hat u_2(\cdot))$ as an *open-loop saddle point* of the game over $[0,T]$ at x, and $V(x)\stackrel{\Delta}{=}J_x(\hat u_1(\cdot),\hat u_2(\cdot))$ as the *open-loop value* of the game at x. We point out that in general, an open-loop saddle point (if it exists) is not necessarily unique. Clearly, when an open-loop saddle point exists for each $x\in\mathbb{R}^n$, the open-loop value function $x\mapsto V(x)$ can be defined. On the other hand, by thinking about an optimal control problem (which can be regarded as a game with the dimension m_2 of the space that $u_2(t)$ takes values being 0), one immediately realizes that, in general, one should not expect to have the existence of an open-loop saddle point from the existence of the open-loop value function. We will return to this point later.

To roughly state our main results, let us introduce the following system of SDE, called forward-backward stochastic differential equation (FBSDE, for short):

$$\begin{cases} dX(t)=\Big[A(t)X(t)+B(t)u(t)\Big]dt+\Big[C(t)X(t)+D(t)u(t)\Big]dW(t),\\ dY(t)=-\Big[Q(t)X(t)+A^\top(t)Y(t)+C^\top(t)Z(t)+S^\top(t)u(t)\Big]dt+Z(t)dW(t),\\ X(0)=x,\qquad Y(T)=GX(T),\\ S(t)X(t)+B^\top(t)Y(t)+D^\top(t)Z(t)+R(t)u(t)=0,\qquad t\in[0,T],\ \text{a.s.}\end{cases} \tag{1.5}$$

The unknown in the above is the triple of $\mathbb{F}$-adapted processes $(X(\cdot),Y(\cdot),Z(\cdot))$. Note that the above is a coupled FBSDE with the coupling through $u(\cdot)$ and the last equality (see [19], [25]). We also introduce the following differential equation called a generalized Riccati equation:

$$\begin{cases} \dot{P} + PA + A^\top P + C^\top PC + Q \\ \qquad -(B^\top P + D^\top PC + S)^\top (D^\top PD + R)^\dagger (B^\top P + D^\top PC + S) = 0, \\ P(T) = G, \\ \Big[I - (D^\top PD + R)(D^\top PD + R)^\dagger\Big](B^\top P + D^\top PC + S) = 0, \\ D_1^\top PD_1 + R_{11} \geq 0, \quad D_2^\top PD_2 + R_{22} \leq 0, \end{cases} \tag{1.6}$$

where $M^\dagger$ stands for the pseudo-inverse of matrix M (see [23]), and

$$B = (B_1, B_2), \quad D = (D_1, D_2), \quad S = \begin{pmatrix} S_1 \\ S_2 \end{pmatrix}, \quad R = \begin{pmatrix} R_{11} & R_{12} \\ R_{21} & R_{22} \end{pmatrix}. \tag{1.7}$$

Note that the third condition in (1.6) is equivalent to the following range condition:

$$\mathcal{R}(B^\top P + D^\top PC + S) \subseteq \mathcal{R}(D^\top PD + R), \tag{1.8}$$

where $\mathcal{R}(M)$ stands for the range of matrix M. For stochastic LQ problems, a similar Riccati differential equation was introduced in [1] in which $D_1 = D$, $R_{11} = R$, and $m_2 = 0$ (thus, the last condition in (1.6) does not appear). The paper [20] discussed some basic properties of generalized Riccati equations arising in stochastic games.

Our main results of this paper can be informally described as follows. The existence of an open-loop saddle point of the game is equivalent to the solvability of FBSDE (1.5) plus the convexity and concavity of $J_x(u_1(\cdot), u_2(\cdot))$ in $u_1(\cdot)$ and $u_2(\cdot)$, respectively. Also, the solvability of (1.6) implies that of FBSDE (1.5). On the other hand, our LQ stochastic differential game can be approached by a leader-follower fashion (see [24]). Namely, for example, let $u_1(\cdot)$ be taken to be the leader and $u_2(\cdot)$ be taken to be the follower. Then for any fixed $u_1(\cdot)$, we first maximize $u_2(\cdot) \mapsto J_x(u_1(\cdot), u_2(\cdot))$. Suppose there exists an optimal control $\hat{u}_2(\cdot) \equiv \hat{u}_2(u_1(\cdot))$ (depending on $u_1(\cdot)$). Then we minimize $u_1(\cdot) \mapsto J_x(u_1(\cdot), \hat{u}_2(u_1(\cdot)))$. Among other things, we show that if such a leader-follower problem admits an optimal solution, it must be an open-loop saddle point of our LQ stochastic differential game.

There is extensive literature on deterministic LQ differential games (i.e., $C(\cdot) = 0$, $D_1(\cdot) = 0$, and $D_2(\cdot) = 0$) and stochastic LQ differential games with the diffusion term independent of the state and controls (see [13], [16], [3], [4], [9], [15], for examples). The case that the controls appear in the diffusion was considered in [24] under a leader-follower framework. For other related works, we mention [6], [14], [7], [8], and [12].

Stochastic LQ control problems can be treated as zero-sum games with only one player. From this point, our results can be considered as generalizations of the relevant results in [10], [16], [1], and [2] for LQ control problems.

The rest of the paper is organized as follows. In Section 2, we present some preliminaries. In Section 3, a careful study of LQ games in Hilbert space is carried out. Section 4 is devoted to the main results of this paper.

2. **Preliminaries.** In this section, we present some preliminaries. First of all, besides $\mathbb{R}^n$, the n-dimensional Euclidean space, and $\mathbb{R}^{n\times m}$, the spaces of all $(n\times m)$ matrices, endowed with the inner product $(M, N) \mapsto \operatorname{tr}[M^\top N]$, we let $\mathcal{S}^n \subseteq \mathbb{R}^{n\times n}$ be the set of all $(n\times n)$ symmetric matrices. Next, for any Euclidean space H (such as $\mathbb{R}^n$, $\mathbb{R}^{n\times m}$, $\mathcal{S}^n$, etc.), and $p \in [1, \infty]$, let $L^p(0, T; H)$ be the set of all L^p-integrable

functions $\varphi : [0,T] \to H$, and $W^{1,p}(0,T;H)$ be the set of all absolutely continuous functions $\varphi : [0,T] \to H$ such that $\dot\varphi(\cdot) \in L^p(0,T;H)$. Further, we keep the basic setting involving $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ and $W(\cdot)$ the same as that in the previous section. Let $L^2_{\mathcal{F}}(0,T;H)$ be the Hilbert space that consists of all $\mathbb{F}$-adapted processes $\varphi(\cdot)$ valued in H such that $\mathbb{E}\int_0^T |\varphi(r)|^2 dr < \infty$ (the inner product of which is denoted by $\langle \cdot\,,\cdot\rangle$), and let $L^2_{\mathcal{F}}(\Omega; C([0,T];H))$ be the Banach space consists of all $\mathbb{F}$-adapted continuous processes $\varphi(\cdot)$ such that $\mathbb{E}\big[\sup_{r\in[0,T]} |\varphi(r)|^2\big] < \infty$.

For $i = 1, 2$, let $\mathcal{U}_i = L^2_{\mathcal{F}}(0,T;\mathbb{R}^{m_i})$. Any process $u_i(\cdot) \in \mathcal{U}_i$ is called an *open-loop control* of Player i on $[0,T]$. Before going further, we need the following standing assumptions which will be assumed in the rest of the paper.

(A1) Let

$$A(\cdot), C(\cdot) \in L^\infty(0,T,\mathbb{R}^{n\times n}), \quad B_i(\cdot), D_i(\cdot) \in L^\infty(0,T;\mathbb{R}^{n\times m_i}), \quad i = 1,2. \tag{2.1}$$

(A2) For $i, j = 1, 2$, let

$$\begin{aligned} &G \in \mathcal{S}^n, \quad Q(\cdot) \in L^\infty(0,T;\mathcal{S}^n), \quad S_i(\cdot) \in L^\infty(0,T;\mathbb{R}^{m_i\times n}),\\ &R_{ii}(\cdot) \in L^\infty(0,T;\,\mathcal{S}^{m_i}), \quad R_{ij}(\cdot) = R_{ji}^\top(\cdot) \in L^\infty(0,T;\,\mathbb{R}^{m_i\times m_j}), \quad i,j = 1,2. \end{aligned} \tag{2.2}$$

In what follows, sometimes, we will denote

$$\begin{aligned} &B(\cdot) = (B_1(\cdot), B_2(\cdot)), \quad D(\cdot) = (D_1(\cdot), D_2(\cdot)),\\ &S = \begin{pmatrix} S_1 \\ S_2 \end{pmatrix}, \quad R(\cdot) = \begin{pmatrix} R_{11}(\cdot) & R_{12}(\cdot) \\ R_{21}(\cdot) & R_{22}(\cdot) \end{pmatrix}. \end{aligned} \tag{2.3}$$

By a standard well-posedness theorem for linear stochastic differential equation (see [27], Chapter 1, Theorem 6.14, for example), under (A1), for any $x \in \mathbb{R}^n$ and $(u_1(\cdot), u_2(\cdot)) \in \mathcal{U}_1 \times \mathcal{U}_2$, state equation (1.1) has a unique strong solution $X(\cdot) \stackrel{\Delta}{=} X(\cdot\,; x, u_1(\cdot), u_2(\cdot)) \in L^2_{\mathcal{F}}(\Omega; C([0,T];\mathbb{R}^n))$. Then, under (A2), $J_x(u_1(\cdot), u_2(\cdot))$ is well-defined, and one can talk about open-loop saddle points and open-loop value functions, and so on. More precisely, we have the following definition.

Definition 2.1. (i) A pair $(\hat u_1(\cdot), \hat u_2(\cdot)) \in \mathcal{U}_1 \times \mathcal{U}_2$ is called an *open-loop saddle point* of the game over $[0,T]$ with respect to $x \in \mathbb{R}^n$ if

$$J_x(\hat u_1(\cdot), u_2(\cdot)) \le J_x(\hat u_1(\cdot), \hat u_2(\cdot)) \le J_x(u_1(\cdot), \hat u_2(\cdot)), \qquad (u_1(\cdot), u_2(\cdot)) \in \mathcal{U}_1 \times \mathcal{U}_2. \tag{2.4}$$

(ii) Let $x \in \mathbb{R}^n$. The *open-loop upper value* of the game at x is defined by

$$V^+(x) = \inf_{u_1(\cdot)\in\mathcal{U}_1} \sup_{u_2(\cdot)\in\mathcal{U}_2} J_x(u_1(\cdot), u_2(\cdot)), \tag{2.5}$$

and the *open-loop lower value* of the game at x is defined by

$$V^-(x) = \sup_{u_2(\cdot)\in\mathcal{U}_2} \inf_{u_1(\cdot)\in\mathcal{U}_1} J_{s,x}(u_1(\cdot), u_2(\cdot)). \tag{2.6}$$

In the case that

$$V^-(x) = V^+(x) \equiv V(x), \tag{2.7}$$

we call $V(x)$ the *open-loop value* of the game at x. Call $V^\pm(\cdot)$ the *open-loop upper* and *lower value functions* of the game, respectively, and call $V(\cdot)$ the *open-loop value function* of the game.

It is easy to see that the existence of an open-loop saddle point implies the existence of the open-loop value, i.e., when (2.4) holds at x, the game has the open-loop value at x and

$$V(x) = J_x(\hat u_1(\cdot), \hat u_2(\cdot)). \tag{2.8}$$

We will see that the converse is not true in general; see Proposition 3.10.

3. **LQ Games in Hilbert Spaces.** We will see in the next section that our two-person zero-sum LQ stochastic differential game can be transformed into a two-person zero-sum LQ games in Hilbert spaces. Hence, in this section, we first look at LQ games in Hilbert spaces.

Let $\mathcal{H}$ be a Hilbert space and $\Theta : \mathcal{D}(\Theta) \subseteq \mathcal{H} \to \mathcal{H}$ be a self-adjoint operator (By definition, it is densely defined and closed, but is not necessarily bounded). We denote $\mathcal{R}(\Theta)$ and $\mathcal{N}(\Theta)$ to be the range and kernel of Θ, respectively. Since Θ is self-adjoint, $\mathcal{N}(\Theta)^\perp = \overline{\mathcal{R}(\Theta)}$ (and we always have $\Theta\Big(\mathcal{D}(\Theta)\bigcap\overline{\mathcal{R}(\Theta)}\Big) \subseteq \mathcal{R}(\Theta)$). Thus, under the decomposition $\mathcal{H} = \mathcal{N}(\Theta) \oplus \overline{\mathcal{R}(\Theta)}$, we have the following representation for Θ:

$$\Theta = \begin{pmatrix} 0 & 0 \\ 0 & \widehat\Theta \end{pmatrix}, \tag{3.1}$$

where $\widehat\Theta : \mathcal{D}(\Theta)\bigcap\overline{\mathcal{R}(\Theta)} \subseteq \overline{\mathcal{R}(\Theta)} \to \overline{\mathcal{R}(\Theta)}$ is self-adjoint (again, it is densely defined and closed, but not necessarily bounded, on the Hilbert space $\overline{\mathcal{R}(\Theta)}$). Now, we define the *pseudo-inverse* $\Theta^\dagger$ by the following:

$$\Theta^\dagger = \begin{pmatrix} 0 & 0 \\ 0 & \widehat\Theta^{-1} \end{pmatrix}, \tag{3.2}$$

with domain

$$\mathcal{D}(\Theta^\dagger) = \mathcal{N}(\Theta) + \mathcal{R}(\Theta) \equiv \{u^0 + u^1 \bigm| u^0 \in \mathcal{N}(\Theta), u^1 \in \mathcal{R}(\Theta)\} \supseteq \mathcal{R}(\Theta). \tag{3.3}$$

From the above, we can easily seen the following facts:

(i) $\Theta^\dagger$ is (closed, densely defined, and) self-adjoint; $\mathcal{R}(\Theta)$ is closed if and only if $\Theta^\dagger$ is bounded.

(ii) By the definition of $\Theta^\dagger$ (see (3.2)), together with (3.3), one has that

$$\Theta\Theta^\dagger\Theta = \Theta, \quad \Theta^\dagger\Theta\Theta^\dagger = \Theta^\dagger, \quad (\Theta^\dagger)^\dagger = \Theta. \tag{3.4}$$

Thus, by (i), $\mathcal{R}(\Theta^\dagger)$ is closed if and only if Θ is bounded.

(iii) Although $\mathcal{D}(\Theta^\dagger)$ is not necessarily closed, the operator $\Theta\Theta^\dagger : \mathcal{D}(\Theta^\dagger) \to \mathcal{H}$ is an orthogonal projection onto $\mathcal{R}(\Theta)$. Thus, we may naturally extend it, still denoted it by itself, to $\overline{\mathcal{D}(\Theta^\dagger)} = \mathcal{H}$. Hence, $\Theta\Theta^\dagger : \mathcal{H} \to \overline{\mathcal{R}(\Theta)} \subseteq \mathcal{H}$ is the orthogonal projection onto $\overline{\mathcal{R}(\Theta)}$. Similarly, we can extend $\Theta^\dagger\Theta$ to be an orthogonal projection from $\mathcal{H}$ onto $\overline{\mathcal{R}(\Theta^\dagger)} = \mathcal{N}(\Theta^\dagger)^\perp = \mathcal{N}(\Theta)^\perp = \overline{\mathcal{R}(\Theta)}$. Therefore, we have

$$\Theta\Theta^\dagger = \Theta^\dagger\Theta \equiv P_{\overline{\mathcal{R}(\Theta)}} \equiv \text{orthogonal projection onto } \overline{\mathcal{R}(\Theta)}. \tag{3.5}$$

Note that when Θ is bounded, $\Theta^\dagger\Theta$ is already an orthogonal projection from $\mathcal{H}$ onto $\mathcal{R}(\Theta^\dagger) = \overline{\mathcal{R}(\Theta)}$.

(iv) The map $\Theta \mapsto \Theta^\dagger$ is not continuous (which can be seen even from one-dimensional case).

Now, let us consider a quadratic functional on $\mathcal{H}$:

$$J(u) = \langle\, \Theta u, u \,\rangle + 2 \,\langle\, v, u \,\rangle, \qquad u \in \mathcal{D}(\Theta) \subseteq \mathcal{H}, \tag{3.6}$$

where $\Theta : \mathcal{D}(\Theta) \subseteq \mathcal{H} \to \mathcal{H}$ is a self-adjoint linear operator and $v \in \mathcal{H}$ is fixed. The following result is concerned with the completing square and critical point(s) of the functional $J(\cdot)$. Note here that we do not assume positive (negative) semi-definite condition on Θ.

Proposition 3.1. (i) There exists a $\hat{u} \in \mathcal{D}(\Theta)$ such that

$$J(u) = \langle \Theta(u - \hat{u}), u - \hat{u} \rangle - \langle \Theta\hat{u}, \hat{u} \rangle, \qquad \forall u \in \mathcal{D}(\Theta), \tag{3.7}$$

if and only if

$$v \in \mathcal{R}(\Theta)\Big(\subseteq \mathcal{D}(\Theta^\dagger)\Big). \tag{3.8}$$

(ii) Any $\hat{u} \in \mathcal{D}(\Theta)$ satisfies (3.7) if and only if it is a solution of the following equation:

$$\Theta\hat{u} + v = 0, \tag{3.9}$$

which is equivalent to the following:

$$\hat{u} = -\Theta^\dagger v + (I - \Theta^\dagger\Theta)\widetilde{v}, \tag{3.10}$$

for some $\widetilde{v} \in \mathcal{H}$ (in particular, $\hat{u} = -\Theta^\dagger v$ is a solution).

(iii) When (3.7) holds, it is necessary that

$$J(u) = \langle \Theta(u - \hat{u}), u - \hat{u} \rangle - \langle \Theta^\dagger v, v \rangle, \qquad \forall u \in \mathcal{D}(\Theta). \tag{3.11}$$

Moreover, $\hat{u}$ is unique if and only if $\mathcal{N}(\Theta) = \{0\}$.

Proof. (i) For any $\hat{u} \in \mathcal{D}(\Theta)$, on has

$$\begin{aligned} J(u) &\equiv \langle \Theta u, u \rangle + 2\langle v, u \rangle \\ &= \langle \Theta(u - \hat{u}), u - \hat{u} \rangle + 2\langle \Theta\hat{u} + v, u \rangle - \langle \Theta\hat{u}, \hat{u} \rangle, \quad \forall u \in \mathcal{D}(\Theta). \end{aligned} \tag{3.12}$$

Hence, there exists a $\hat{u} \in \mathcal{D}(\Theta)$ such that (3.7) holds if and only if (3.9) holds, which gives (3.8) (and the first part of (ii)).

Conversely, if (3.8) holds, then there exists a $\hat{u} \in \mathcal{D}(\Theta)$ such that (3.9) holds. Consequently,

$$\begin{aligned} \langle \Theta(u - \hat{u}), u - \hat{u} \rangle - \langle \Theta\hat{u}, \hat{u} \rangle &= \langle \Theta u, u \rangle - 2\langle \Theta\hat{u}, u \rangle + \langle \Theta\hat{u}, \hat{u} \rangle - \langle \Theta\hat{u}, \hat{u} \rangle \\ &= \langle \Theta u, u \rangle + 2\langle v, u \rangle = J(u), \end{aligned} \tag{3.13}$$

proving (3.7).

(ii) We have proved the first part of (ii) (from (3.12)). The second part is straightforward.

(iii) For any $\hat{u} \in \mathcal{D}(\Theta)$ satisfying (3.7), one must have (3.9). Hence,

$$\begin{aligned} \langle \Theta(u - \hat{u}), u - \hat{u} \rangle - \langle \Theta^\dagger v, v \rangle &= \langle \Theta u, u \rangle - 2\langle \Theta\hat{u}, u \rangle + \langle \Theta\hat{u}, \hat{u} \rangle - \langle \Theta^\dagger\Theta\hat{u}, \Theta\hat{u} \rangle \\ &= \langle \Theta u, u \rangle + 2\langle v, u \rangle = J(u), \end{aligned} \tag{3.14}$$

which proves (3.11). Finally, by (3.9), we see that $\hat{u}$ is unique if and only if $\mathcal{N}(\Theta) = \{0\}$. □

Note that (3.9) is equivalent to the following:

$$0 = \Theta\hat{u} + v \equiv \frac{1}{2}\nabla J(\hat{u}). \tag{3.15}$$

Thus, $\hat{u}$ is actually a *critical point* of functional $J(\cdot)$. Thus, Proposition 3.1 characterizes the critical point(s) of the quadratic functional $J(\cdot)$. Equations (3.7) and

(3.11) are completion of square for the functional $J(\cdot)$ (although Θ is not necessarily positive/negative semi-definite).

Next, for any self-adjoint operator Θ, regardless whether it is bounded or unbounded, we have the following spectrum decomposition ([11])

$$\Theta = \int_{\sigma(\Theta)} \lambda dP_\lambda, \tag{3.16}$$

where $\sigma(\Theta) \subseteq \mathbb{R}$ is the spectrum of Θ, (which is a compact set if Θ is bounded, and unbounded if Θ is unbounded); and $\{P_\lambda \bigm| \lambda \in \sigma(\Theta)\}$ is a family of projection measures. In the case that

$$\Theta \geq 0, \tag{3.17}$$

one has from (3.16) that $\sigma(\Theta) \subseteq [0, \infty)$, and

$$\begin{cases} \Theta^\alpha = \displaystyle\int_{\sigma(\Theta)} \lambda^\alpha dP_\lambda, \qquad \forall \alpha \geq 0, \\ (\Theta^\dagger)^\alpha = (\Theta^\alpha)^\dagger = \displaystyle\int_{\sigma(\Theta)\setminus\{0\}} \lambda^{-\alpha} dP_\lambda, \qquad \forall \alpha > 0. \end{cases} \tag{3.18}$$

Now, we can consider minimization problem for functional $J(\cdot)$.

Proposition 3.2. *Let $\Theta : \mathcal{D}(\Theta) \subseteq \mathcal{H} \to \mathcal{H}$ be self-adjoint and $v \in \mathcal{H}$.*

(i) The following holds:

$$\inf_{u \in \mathcal{D}(\Theta)} J(u) > -\infty \tag{3.19}$$

if and only if (3.17) holds and

$$v \in \mathcal{R}(\Theta^{\frac{1}{2}}). \tag{3.20}$$

In this case,

$$\inf_{u \in \mathcal{D}(\Theta)} J(u) = -|(\Theta^\dagger)^{\frac{1}{2}} v|^2. \tag{3.21}$$

(ii) There exists a $\hat{u} \in \mathcal{H}$ such that

$$J(\hat{u}) = \inf_{u \in \mathcal{D}(\Theta)} J(u), \tag{3.22}$$

if and only if (3.17) and (3.8) hold; and in this case, all the conclusions in Proposition 3.1 hold.

Proof. (i) First, let (3.19) hold. It is straightforward that one must have (3.17). Next, we prove (3.20) by contradiction. Suppose (3.20) does not hold. For any $n \geq 1$, let

$$v_n = \int_{\sigma(\Theta)\cap[\frac{1}{n}, n]} dP_\lambda v.$$

Then $v_n \in \mathcal{R}(\Theta)$, and

$$\langle\, v, \Theta^\dagger v_n \,\rangle = \int_{\sigma(\Theta)\cap[\frac{1}{n}, n]} \lambda^{-1} d|P_\lambda v|^2 = |(\Theta^\dagger)^{\frac{1}{2}} v_n|^2 \to \infty, \quad n \to \infty.$$

Hence, letting $u_n = -\Theta^\dagger v_n$, we obtain

$$J(u_n) = \langle\, \Theta u_n, u_n \,\rangle + 2\,\langle\, v, u_n \,\rangle = -|(\Theta^\dagger)^{\frac{1}{2}} v_n|^2 \to -\infty, \quad n \to \infty,$$

contradicting (3.19).

Conversely, if (3.17) and (3.20) hold, then for any $u \in \mathcal{D}(\Theta)$, one has

$$\begin{aligned} J(u) &= |\Theta^{\frac{1}{2}} u|^2 + 2\,\langle (\Theta^\dagger)^{\frac{1}{2}} v, \Theta^{\frac{1}{2}} u \,\rangle = |\Theta^{\frac{1}{2}} u + (\Theta^\dagger)^{\frac{1}{2}} v|^2 - |(\Theta^\dagger)^{\frac{1}{2}} v|^2 \\ &\geq -|(\Theta^\dagger)^{\frac{1}{2}} v|^2 > -\infty. \end{aligned} \tag{3.23}$$

Hence, sufficiency follows.

Finally, from the fact that

$$\mathcal{R}((\Theta^\dagger)^{\frac{1}{2}}) \subseteq \overline{\mathcal{R}(\Theta^{\frac{1}{2}})} = \overline{\mathcal{R}(\Theta)},$$

we can always find a sequence $u_n \in \mathcal{D}(\Theta)$ so that (note (3.23))

$$J(u_n) = |\Theta^{\frac{1}{2}} u_n + (\Theta^\dagger)^{\frac{1}{2}} v|^2 - |(\Theta^\dagger)^{\frac{1}{2}} v|^2 \to -|(\Theta^\dagger)^{\frac{1}{2}} v|^2, \quad n \to \infty.$$

Thus, (3.21) follows.

(ii) By Proposition 3.1 (i), we know that (3.8) holds if and only if (3.7) holds for some $\hat{u} \in \mathcal{D}(\Theta)$. Then (3.17) holds if and only if $\hat{u}$ is a minimum. The rest is clear. □

The above result tells us that the existence of minimum is strictly stronger than the finiteness of the infimum of the functional $J(\cdot)$, which have been described by conditions (3.8) and (3.20), respectively. Note here that $\mathcal{R}(\Theta) \subseteq \mathcal{R}(\Theta^{\frac{1}{2}})$ when (3.17) holds.

The following example shows the necessity of condition (3.20) in a concrete way.

Example 3.3. Let $\mathcal{H} = \ell^2$. For any $u = \{a_i\}_{i=1}^\infty \in \mathcal{H}$, define Θu by

$$\Theta u = \{\beta^{i-1} a_i\}_{i=1}^\infty,$$

where $\beta \in (0,1)$. Then $\Theta : \mathcal{H} \to \mathcal{H}$ is bounded, self-adjoint, and positive definite (but not uniformly). Clearly, for any $\alpha \in \mathbb{R}$,

$$\Theta^\alpha u = \{\beta^{\alpha(i-1)} a_i\}_{i=1}^\infty, \qquad \forall u \equiv \{a_i\}_{i=1}^\infty \in \mathcal{H}.$$

Let $v = \{i^{-1}\}_{i=1}^\infty \in \mathcal{H}$. Then $v \in \overline{\mathcal{R}(\Theta)}$ since $v = \lim_{n\to\infty} \Theta u_n$ with

$$u_n = \{1, \frac{1}{2\beta}, \frac{1}{3\beta^2}, \cdots, \frac{1}{n\beta^{n-1}}, 0, 0, \cdots\} \in \mathcal{H}.$$

But, clearly, $v \notin \mathcal{R}(\Theta^\alpha)$ for any $\alpha > 0$ (in particular, $v \notin \mathcal{R}(\Theta^{\frac{1}{2}})$). Now, consider the quadratic functional

$$J(u) = \langle \Theta u, u \rangle + 2 \langle v, u \rangle = \sum_{i=1}^\infty \Big(\beta^{i-1} a_i^2 + \frac{2a_i}{i} \Big).$$

Then by letting u_n as above, we see that

$$\begin{aligned} J(-u_n) &= \langle \Theta u_n, u_n \rangle - 2 \langle v, u_n \rangle \\ &= \sum_{i=1}^n \Big[\beta^{i-1} \frac{1}{i^2 \beta^{2(i-1)}} - \frac{2}{i^2 \beta^{i-1}} \Big] = -\sum_{i=1}^n \frac{1}{i^2 \beta^{(i-1)}} \to -\infty, \quad \text{as} \quad n \to \infty. \end{aligned}$$

This means that

$$\inf_{u \in \mathcal{H}} J(u) = -\infty.$$

An interesting point here is that positive semi-definiteness of Θ is not enough to ensure the finiteness of the infimum of $J(\cdot)$.

In the rest of this section, we let $\mathcal{H} = \mathcal{H}_1 \times \mathcal{H}_2$ with $\mathcal{H}_1$ and $\mathcal{H}_2$ being two Hilbert spaces, and consider a quadratic functional on $\mathcal{H}$:

$$\begin{aligned} J(u) &\equiv J(u_1, u_2) = \langle \Theta u, u \rangle + 2 \langle v, u \rangle \\ &\equiv \Big\langle \begin{pmatrix} \Theta_{11} & \Theta_{12} \\ \Theta_{21} & \Theta_{22} \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}, \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \Big\rangle + 2 \Big\langle \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}, \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \Big\rangle, \\ &\quad \forall u \equiv (u_1, u_2) \in \mathcal{H}. \end{aligned} \tag{3.24}$$

We assume that $\Theta_{ij} : \mathcal{H}_j \to \mathcal{H}_i$ are bounded, $\Theta \equiv \begin{pmatrix} \Theta_{11} & \Theta_{12} \\ \Theta_{21} & \Theta_{22} \end{pmatrix}$ is self-adjoint. Consider a two-person zero-sum game with the cost/payoff functional given by (3.24). In the game, Player 1 takes $u_1 \in \mathcal{H}_1$ to minimize $J(u_1, u_2)$ and Player 2 takes $u_2 \in \mathcal{H}_2$ to maximize $J(u_1, u_2)$. Note that if we take $\mathcal{H} = \mathcal{H}_1$ (i.e., $\mathcal{H}_2 = \{0\}$), then the game problem becomes the minimization problem for a quadratic functional in a Hilbert space. Hence, minimization/maximization problem(s) can be regarded as a special case of zero-sum games. The following result is a natural extension of Proposition 3.2,(ii).

Proposition 3.4. There exists a saddle point $(\hat{u}_1, \hat{u}_2) \in \mathcal{H}_1 \times \mathcal{H}_2$ for $(u_1, u_2) \mapsto J(u_1, u_2)$, that is,

$$J(\hat{u}_1, u_2) \le J(\hat{u}_1, \hat{u}_2) \le J(u_1, \hat{u}_2), \quad \forall (u_1, u_2) \in \mathcal{H}_1 \times \mathcal{H}_2, \tag{3.25}$$

if and only if (3.8) holds and the following are true:

$$\Theta_{11} \ge 0, \qquad \Theta_{22} \le 0. \tag{3.26}$$

In the above case, each saddle point $\hat{u} = (\hat{u}_1, \hat{u}_2) \in \mathcal{H}_1 \times \mathcal{H}_2$ is a solution of the equation (3.9), and it admits a representation (3.10). Moreover, $\hat{u}$ is unique if and only if $\mathcal{N}(\Theta) = \{0\}$.

Proof. Necessity: We first show (3.26) by a contradiction argument. If $\Theta_{11} \ge 0$ is not true, then $\langle \Theta_{11} u_1, u_1 \rangle < 0$, for some $u_1 \in \mathcal{H}_1$. Consequently, we have that (note (3.25))

$$J(\hat{u}_1, \hat{u}_2) \le \lim_{\lambda \to \infty} J(\lambda u_1, \hat{u}_2) = \lim_{\lambda \to \infty} \lambda^2 J(u_1, \hat{u}_2/\lambda) = \lim_{\lambda \to \infty} \lambda^2 \langle \Theta_{11} u_1, u_1 \rangle = -\infty. \tag{3.27}$$

This is a contradiction. Hence, $\Theta_{11} \ge 0$ must be true. Similarly, $\Theta_{22} \le 0$ must hold.

Next, by (3.12) and (3.25), we have

$$\langle \Theta_{22}(u_2 - \hat{u}_2), u_2 - \hat{u}_2 \rangle + 2 \langle \Theta_{21}\hat{u}_1 + \Theta_{22}\hat{u}_2 + v_2, u_2 - \hat{u}_2 \rangle = J(\hat{u}_1, u_2) - J(\hat{u}_1, \hat{u}_2) \le 0, \tag{3.28}$$

for all $u_2 \in \mathcal{H}_2$. Hence, it is necessary that

$$\Theta_{21}\hat{u}_1 + \Theta_{22}\hat{u}_2 + v_2 = 0. \tag{3.29}$$

Similarly,

$$\Theta_{11}\hat{u}_1 + \Theta_{12}\hat{u}_2 + v_1 = 0. \tag{3.30}$$

Thus, (3.8) follows.

Sufficiency: Let (3.8) hold. Then map $(u_1, u_2) \mapsto J(u_1, u_2)$ admits a critical point $\hat{u} \equiv (\hat{u}_1, \hat{u}_2)$. By Proposition 3.1, we get

$$J(u_1, u_2) = \langle \Theta(u - \hat{u}), u - \hat{u} \rangle - \langle \Theta^{\dagger} v, v \rangle . \tag{3.31}$$

Thus, $J(\hat{u}_1, \hat{u}_2) = -\langle \Theta^{\dagger} v, v \rangle$. Since $\Theta_{11} \ge 0$ and $\Theta_{22} \le 0$, it follows that

$$\begin{aligned} J(\hat{u}_1, u_2) &= J(\hat{u}_1, \hat{u}_2) + \langle \Theta_{22}(u_2 - \hat{u}_2), u_2 - \hat{u}_2 \rangle \le J(\hat{u}_1, \hat{u}_2), \\ J(u_1, \hat{u}_2) &= J(\hat{u}_1, \hat{u}_2) + \langle \Theta_{11}(u_1 - \hat{u}_1), u_1 - \hat{u}_1 \rangle \ge J(\hat{u}_1, \hat{u}_2) . \end{aligned} \tag{3.32}$$

Hence, (3.25) follows.

The rest of the proof is clear. □

We have the following interesting corollary.

Corollary 3.5. Suppose that (3.26) holds and Θ_{11}^{-1} and Θ_{22}^{-1} exist; Or equivalently, suppose that $J(u_1,0)$ admits a unique minimizer and $J(0,u_2)$ admits a unique maximizer. Then the game admits a unique saddle point.

Proof. When (3.26) holds, Θ_{11}^{-1} and Θ_{22}^{-1} exist, we can directly check that Θ^{-1} exists and is bounded. Hence, Proposition 3.4 applies. By Proposition 3.2 (ii), we know that the above-mentioned conditions are equivalent to the unique existence of the minimizer and the maximizer of $J(u_1,0)$ and $J(0,u_2)$, respectively. □

The above result tells us that for quadratic functionals, the existence of a unique saddle point is guaranteed by the existence of solutions to two optimization problems for quadratic functionals. But, the above result does not give a construction of the saddle point. The following result gives a construction of a saddle point.

Corollary 3.6. Suppose that (3.26) holds.

(i) Suppose the following holds:

$$v_1 \in \mathcal{R}(\Theta_{11}), \qquad v_2 - \Theta_{21}\Theta_{11}^{\dagger}v_1 \in \mathcal{R}(\Theta_{22} - \Theta_{21}\Theta_{11}^{\dagger}\Theta_{12}). \tag{3.33}$$

Then the game admits a saddle point given by the following:

$$\begin{cases} u_1 = -\Theta_{11}^{\dagger}[v_1 - \Theta_{12}(\Theta_{22} - \Theta_{21}\Theta_{11}^{\dagger}\Theta_{12})^{\dagger}(v_2 - \Theta_{21}\Theta_{11}^{\dagger}v_1)], \\ u_2 = -(\Theta_{22} - \Theta_{21}\Theta_{11}^{\dagger}\Theta_{12})^{\dagger}(v_2 - \Theta_{21}\Theta_{11}^{\dagger}v_1). \end{cases} \tag{3.34}$$

(ii) Suppose the following holds:

$$v_2 \in \mathcal{R}(\Theta_{22}), \qquad v_1 - \Theta_{12}\Theta_{22}^{\dagger}v_2 \in \mathcal{R}(\Theta_{11} - \Theta_{12}\Theta_{22}^{\dagger}\Theta_{21}). \tag{3.35}$$

Then the game admits a saddle point given by the following:

$$\begin{cases} u_1 = -(\Theta_{11} - \Theta_{12}\Theta_{22}^{\dagger}\Theta_{21})^{\dagger}(v_1 - \Theta_{12}\Theta_{22}^{\dagger}v_2), \\ u_2 = -\Theta_{22}^{\dagger}[v_2 - \Theta_{21}(\Theta_{11} - \Theta_{12}\Theta_{22}^{\dagger}\Theta_{21})^{\dagger}(v_1 - \Theta_{12}\Theta_{22}^{\dagger}v_2)]. \end{cases} \tag{3.36}$$

(iii) The saddle points given in (3.34) and (3.36) are the same if either

$$\Theta_{11}\Theta_{11}^{\dagger}\Theta_{12} = \Theta_{12}, \tag{3.37}$$

or

$$\Theta_{22}\Theta_{22}^{\dagger}\Theta_{21} = \Theta_{21}. \tag{3.38}$$

Proof. (i) Let (3.33) hold. Let (u_1,u_2) be defined by (3.34). One can directly check that such a pair (u_1,u_2) is a solution of (3.9). Hence, by Proposition 3.4, the game admits a saddle point. The proof of (ii) is similar.

To prove (iii), we show that the saddle points in both (3.34) and (3.36) are $-\Theta^{\dagger}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$. As an example, we prove that this is the case for (3.34) under condition (3.37). The proof for the case of (3.36) under condition (3.38) is similar. Write (3.34) as $\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = -M\begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$, where

$$M = \begin{pmatrix} \Theta_{11}^{\dagger} + \Theta_{11}^{\dagger}\Theta_{12}\Psi^{\dagger}\Theta_{21}\Theta_{11}^{\dagger} & -\Theta_{11}^{\dagger}\Theta_{12}\Psi^{\dagger} \\ -\Psi^{\dagger}\Theta_{21}\Theta_{11}^{\dagger} & \Psi^{\dagger} \end{pmatrix} \tag{3.39}$$

and $\Psi = \Theta_{22} - \Theta_{21}\Theta_{11}^{\dagger}\Theta_{12}$. We need only to show that $\Theta^{\dagger} = M$. Note that M defined above is self-adjoint. Using that $\Theta_{11}\Theta_{11}^{\dagger}\Theta_{12} = \Theta_{12}$ and (3.4), we have

$$\begin{aligned}
\Theta M &= \begin{pmatrix} \Theta_{11}\Theta_{11}^{\dagger} & 0 \\ (I-\Psi\Psi^{\dagger})\Theta_{21}\Theta_{11}^{\dagger} & \Psi\Psi^{\dagger} \end{pmatrix}, \\
\Theta M\Theta &= \begin{pmatrix} \Theta_{11} & \Theta_{12} \\ (I-\Psi\Psi^{\dagger})\Theta_{21} + \Psi\Psi^{\dagger}\Theta_{21} & (I-\Psi\Psi^{\dagger})\Theta_{21}\Theta_{11}^{\dagger}\Theta_{12} + \Psi\Psi^{\dagger}\Theta_{22} \end{pmatrix} \\
&= \begin{pmatrix} \Theta_{11} & \Theta_{12} \\ \Theta_{21} & (I-\Psi\Psi^{\dagger})(\Theta_{22}-\Psi) + \Psi\Psi^{\dagger}\Theta_{22} \end{pmatrix} = \Theta, \\
M\Theta M &= M.
\end{aligned} \tag{3.40}$$

Since $\Theta^{\dagger}$ is the unique self-adjoint operator M satisfying $M\Theta M = M$ and $\Theta M\Theta = \Theta$, we have $\Theta^{\dagger} = M$. □

One can approach the two-person zero-sum game in a leader-follower fashion. More precisely, suppose Player 2 is the leader and Player 1 is the follower. First, the follower minimizes his/her cost functional $u_1 \mapsto J(u_1, u_2)$ for any leader's control u_2. Then the leader wants to maximizes his/her payoff functional $J_2(u_2) = \inf_{u_1\in\mathcal{H}_1} J(u_1, u_2)$. One can reverse the role of Players 1 and 2. We refer to the above as a leader-follower game (or an iterative optimization problem). A natural question is whether we can obtain a saddle point of the original two-person zero-sum game by solving a leader-follower game? The following result gives a positive answer, under certain conditions.

Corollary 3.7. Let (3.26) hold.

(i) Suppose for any $u_2 \in \mathcal{H}_2$, there exists a $\bar{u}_1(u_2) \in \mathcal{H}_1$ such that

$$J(\bar{u}_1(u_2), u_2) = \inf_{u_1\in\mathcal{H}_1} J(u_1, u_2), \tag{3.41}$$

and there exists a $\hat{u}_2 \in \mathcal{H}_2$ such that

$$J(\bar{u}_1(\hat{u}_2), \hat{u}_2) = \sup_{u_2\in\mathcal{H}_2} J(\bar{u}_1(u_2), u_2). \tag{3.42}$$

Then $(\hat{u}_1, \hat{u}_2) \stackrel{\Delta}{=} (\bar{u}_1(\hat{u}_2), \hat{u}_2) \in \mathcal{H}_1 \times \mathcal{H}_2$ is a saddle point of the game and (3.34) gives such a saddle point.

(ii) Suppose for any $u_1 \in \mathcal{H}_1$, there exists a $\bar{u}_2(u_1) \in \mathcal{H}_2$ such that

$$J(u_1, \bar{u}_2(u_1)) = \sup_{u_2\in\mathcal{H}_2} J(u_1, u_2), \tag{3.43}$$

and there exists a $\tilde{u}_1 \in \mathcal{H}_1$ such that

$$J(\tilde{u}_1, \bar{u}_2(\tilde{u}_1)) = \inf_{u_1\in\mathcal{H}_1} J(u_1, \bar{u}_2(u_1)). \tag{3.44}$$

Then $(\tilde{u}_1, \tilde{u}_2) \stackrel{\Delta}{=} (\tilde{u}_1, \bar{u}_2(\tilde{u}_1)) \in \mathcal{H}_1 \times \mathcal{H}_2$ is a saddle point of the game and (3.36) gives such a saddle point.

(iii) The saddle points given in (3.34) and (3.36) are both $-\Theta^{\dagger}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$, which is a saddle point defined in both (i) and (ii).

Proof. We prove (i) and (iii) only. For any $u_2 \in \mathcal{H}_2$, let us consider the minimization problem for the functional $u_1 \mapsto J(u_1, u_2)$. Recall that

$$J(u_1, u_2) = \langle \Theta_{11}u_1, u_1 \rangle + 2\langle v_1 + \Theta_{12}u_2, u_1 \rangle + \langle \Theta_{22}u_2, u_2 \rangle + 2\langle v_2, u_2 \rangle. \tag{3.45}$$

Thus, by Proposition 3.2, there exists a $\bar u_1(u_2)$ such that (3.41) holds if and only if

$$\Theta_{11}\ge 0,\qquad v_1+\Theta_{12}u_2\in\mathcal{R}(\Theta_{11}) \tag{3.46}$$

and a minimizer is given by

$$\bar u_1(u_2)=-\Theta_{11}^{\dagger}(v_1+\Theta_{12}u_2), \tag{3.47}$$

which satisfies $\Theta_{11}\bar u_1(u_2)+\Theta_{12}u_2+v_1=0$. Since u_2 is arbitrary,

$$v_1\in\mathcal{R}(\Theta_{11}),\qquad \Theta_{11}\Theta_{11}^{\dagger}\Theta_{12}=\Theta_{12}. \tag{3.48}$$

Furthermore, we have

$$\begin{aligned} J_2(u_2)&\equiv \inf_{u_1\in\mathcal{H}_1}J(u_1,u_2)=-|(\Theta_{11}^{\dagger})^{\frac12}(v_1+\Theta_{12}u_2)|^2+\langle\,\Theta_{22}u_2,u_2\,\rangle+2\,\langle\, v_2,u_2\,\rangle\\ &=\langle(\Theta_{22}-\Theta_{21}\Theta_{11}^{\dagger}\Theta_{12})u_2,u_2\,\rangle+2\,\langle\, v_2-\Theta_{21}\Theta_{11}^{\dagger}v_1,u_2\,\rangle-|(\Theta_{11}^{\dagger})^{\frac12}v_1|^2. \end{aligned} \tag{3.49}$$

Now, by Proposition 3.2 again, we see that there exists a $\hat u_2$ that maximizes $J_2(\cdot)$ if and only if

$$\Theta_{22}-\Theta_{21}\Theta_{11}^{\dagger}\Theta_{12}\le 0,\qquad v_2-\Theta_{21}\Theta_{11}^{\dagger}v_1\in\mathcal{R}(\Theta_{22}-\Theta_{21}\Theta_{11}^{\dagger}\Theta_{12}),$$

and a maximizer is given by

$$\hat u_2=-(\Theta_{22}-\Theta_{21}\Theta_{11}^{\dagger}\Theta_{12})^{\dagger}(v_2-\Theta_{21}\Theta_{11}^{\dagger}v_1). \tag{3.50}$$

Note that the point $\begin{pmatrix}\hat u_1\\ \hat u_2\end{pmatrix}\triangleq\begin{pmatrix}\bar u_1(\hat u_2)\\ \hat u_2\end{pmatrix}$ defined by (3.47) and (3.50) is exactly the same as given in (3.34), which must be a saddle point of the game by Proposition 3.4 (or Corollary 3.6(i)).

Similarly, the point defined in (3.36) is a saddle point $\begin{pmatrix}\tilde u_1\\ \tilde u_2\end{pmatrix}$ in case (ii). In particular, $\Theta_{22}\Theta_{22}^{\dagger}\Theta_{21}=\Theta_{21}$ holds in this case. In other words, both conditions (3.37) and (3.38) in Corollary 3.6 (iii) hold. Therefore, $\begin{pmatrix}\hat u_1\\ \hat u_2\end{pmatrix}=\begin{pmatrix}\tilde u_1\\ \tilde u_2\end{pmatrix}=-\Theta^{\dagger}\begin{pmatrix}v_1\\ v_2\end{pmatrix}$. This proves (iii). $\square$

We should point out that the conditions imposed in the above result are sufficient for the existence of a saddle point, and they are not necessary.

From Proposition 3.2 (i), we know that for a minimization problem (which is a special case of games), finiteness of the infimum of the functional does not necessarily imply the existence of a minimizer. In the general game case, we expect to have a similar situation. To be more precise, we introduce the upper value V^+ and the lower value V^- of the game as follows:

$$V^+\triangleq\inf_{u_1\in\mathcal{H}_1}\sup_{u_2\in\mathcal{H}_2}J(u_1,u_2),\qquad V^-\triangleq\sup_{u_2\in\mathcal{H}_2}\inf_{u_1\in\mathcal{H}_1}J(u_1,u_2). \tag{3.51}$$

In general, we have

$$V^-\le V^+, \tag{3.52}$$

regardless if V^+ and/or V^- are finite or infinite. If both $V^\pm$ are finite and they are equal, we say that the game has a value. It is easy to show that if $J(u_1,u_2)$ defined by (3.24) admits a saddle point $(\hat u_1,\hat u_2)$, then

$$J(\hat u_1,\hat u_2)=V^+=V^-. \tag{3.53}$$

This means that the existence of a saddle point implies the existence of the value. On the other hand, as we mentioned above, the existence of the value does not necessarily imply the existence of a saddle point. We will see a more delicate case a little later.

The following proposition collects some results on the upper and lower values for the game.

Proposition 3.8. Among the following statements, it holds (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii):

(i) The game has a value, i.e., $V^- = V^+$;

(ii) Both upper and lower values $V^\pm$ are finite;

(iii) Condition (3.26) holds and

$$v \in \overline{\mathcal{R}(\Theta)}\,. \tag{3.54}$$

Proof. (i) $\Rightarrow$ (ii) is trivial.

(ii) $\Rightarrow$ (iii): We first show (3.26) by a contradiction argument. If $\Theta_{11} \geq 0$ is not true, then $\langle\, \Theta_{11}u_1, u_1 \,\rangle < 0$, for some $u_1 \in \mathcal{H}_1$. Consequently, for any $u_2 \in \mathcal{H}_2$, we have that

$$\lim_{\lambda\to\infty} J(\lambda u_1, u_2) = \lim_{\lambda\to\infty} \lambda^2 J(u_1, u_2/\lambda) = \lim_{\lambda\to\infty} \lambda^2 \langle\, \Theta_{11}u_1, u_1 \,\rangle = -\infty.$$

This contradicts the finiteness of V^-. Hence, $\Theta_{11} \geq 0$ must be true. Similarly, by the finiteness of V^+, $\Theta_{22} \leq 0$ holds.

Now we show that $v \in \overline{\mathcal{R}(\Theta)} = \mathcal{N}(\Theta)^\perp$. Let $\hat u \in \mathcal{N}(\Theta)$, that is,

$$\Theta \hat u \equiv \begin{pmatrix} \Theta_{11} & \Theta_{12} \\ \Theta_{21} & \Theta_{22} \end{pmatrix} \begin{pmatrix} \hat u_1 \\ \hat u_2 \end{pmatrix} = \begin{pmatrix} \Theta_{11}\hat u_1 + \Theta_{12}\hat u_2 \\ \Theta_{21}\hat u_1 + \Theta_{22}u_2 \end{pmatrix} = 0. \tag{3.55}$$

We want to show that $\langle\, v, \hat u \,\rangle = 0$. To this end, we note that $\Theta_{12} = \Theta_{21}^*$. Hence, by (3.55), one has

$$\langle\, \Theta_{11}\hat u_1, \hat u_1 \,\rangle = -\langle\, \Theta_{12}\hat u_2, \hat u_1 \,\rangle = -\langle\, \hat u_2, \Theta_{21}\hat u_1 \,\rangle = \langle\, \Theta_{22}\hat u_2, \hat u_2 \,\rangle\,.$$

Due to (3.26), we must have

$$\Theta_{11}\hat u_1 = 0, \qquad \Theta_{22}\hat u_2 = 0.$$

Hence, it follows from (3.55) that

$$\Theta_{12}\hat u_2 = 0, \qquad \Theta_{21}\hat u_1 = 0.$$

Consequently,

$$J(\lambda \hat u_1, u_2) = 2\lambda \langle\, v_1, \hat u_1 \,\rangle + 2\langle\, v_2, u_2 \,\rangle + \langle\, \Theta_{22}u_2, u_2 \,\rangle\,. \tag{3.56}$$

By the finiteness of V^-, we can find some $\bar u_2 \in \mathcal{H}_2$ such that

$$-\infty < \inf_{u_1\in\mathcal{H}_1} J(u_1, \bar u_2) \leq \inf_{\lambda\in\mathbb{R}} J(\lambda \hat u_1, \bar u_2).$$

Hence, we must have $\langle\, v_1, \hat u_1 \,\rangle = 0$. Similarly, one can obtain $\langle\, v_2, \hat u_2 \,\rangle = 0$. These imply $\langle\, v, \hat u \,\rangle = 0$, proving (iii). $\square$

Since an optimization problem is a special case of the game, by Proposition 3.2, we know that in the above proposition, (iii) does not necessarily imply (ii) (see Example 3.3 also). It is not clear if (i) and (ii) in the above proposition are equivalent. This amounts to asking if the finiteness of both upper and lower values necessarily implies the existence of the value.

Remark 3.9. 1) Combining Propositions 3.4 and 3.8, we see that if Θ has a closed range, which is the case if Θ^{-1} exists and is bounded, or if both $\mathcal{H}_1$ and $\mathcal{H}_2$ are finite-dimensional, then (i)–(iii) in Proposition 3.8 are all equivalent and they are also equivalent to the existence of a saddle point.

2) Condition (3.26) is equivalent to the fact that $u_1 \mapsto J(u_1, u_2)$ is convex and $u_2 \mapsto J(u_1, u_2)$ is concave. Hence, the convexity of $J(u_1, u_2)$ in u_1 and concavity of $J(u_1, u_2)$ in u_2 are necessary conditions for the game to have finite upper and lower values. By Corollary 3.5, if the above-mentioned convexity and concavity are uniformly strict (which means that (3.26) holds and Θ_{11}^{-1} and Θ_{22}^{-1} exist and are bounded), then the game admits a unique saddle point. However, Example 3.3 shows that condition (3.26), together with (3.54), does not even necessarily guarantee the finiteness of the upper or lower value.

The following proposition gives some sufficient conditions under which V^+, V^-, or both are finite, respectively, under conditions which are weaker than those for the existence of a saddle point.

Proposition 3.10. (i) If the following holds:

$$\begin{cases} \Theta_{22} \le 0, \qquad v_2 \in \mathcal{R}(\Theta_{22}), \\ \Theta_{11} - \Theta_{12}\Theta_{22}^{\dagger}\Theta_{21} \ge 0, \qquad v_1 - \Theta_{12}\Theta_{22}^{\dagger}v_2 \in \mathcal{R}\Big((\Theta_{11} - \Theta_{12}\Theta_{22}^{\dagger}\Theta_{21})^{\frac{1}{2}}\Big), \end{cases} \tag{3.57}$$

then V^+ is finite and is given by

$$V^+ = -\Big|[(\Theta_{11} - \Theta_{12}\Theta_{22}^{\dagger}\Theta_{21})^{\dagger}]^{\frac{1}{2}}(v_1 - \Theta_{12}\Theta_{22}^{\dagger}v_2)\Big|^2 + \Big|(-\Theta_{22}^{\dagger})^{\frac{1}{2}}v_2\Big|^2. \tag{3.58}$$

(ii) If the following holds:

$$\begin{cases} \Theta_{11} \ge 0, \qquad v_1 \in \mathcal{R}(\Theta_{11}), \\ \Theta_{21}\Theta_{11}^{\dagger}\Theta_{12} - \Theta_{22} \ge 0, \qquad v_2 - \Theta_{21}\Theta_{11}^{\dagger}v_1 \in \mathcal{R}\Big((\Theta_{21}\Theta_{11}^{\dagger}\Theta_{12} - \Theta_{22})^{\frac{1}{2}}\Big), \end{cases} \tag{3.59}$$

then V^- is finite and is given by

$$V^- = \Big|[(\Theta_{21}\Theta_{11}^{\dagger}\Theta_{12} - \Theta_{22})^{\dagger}]^{\frac{1}{2}}(v_2 - \Theta_{21}\Theta_{11}^{\dagger}v_1)\Big|^2 - \Big|(\Theta_{11}^{\dagger})^{\frac{1}{2}}v_1\Big|^2. \tag{3.60}$$

(iii) If the following holds:

$$\begin{cases} \Theta_{11} \ge 0, \qquad \Theta_{22} \le 0, \\ v_1 \in \mathcal{R}(\Theta_{11}), \qquad v_2 \in \mathcal{R}(\Theta_{22}), \\ v_1 - \Theta_{12}\Theta_{22}^{\dagger}v_2 \in \mathcal{R}\Big((\Theta_{11} - \Theta_{12}\Theta_{22}^{\dagger}\Theta_{21})^{\frac{1}{2}}\Big), \\ v_2 - \Theta_{21}\Theta_{11}^{\dagger}v_1 \in \mathcal{R}\Big((\Theta_{21}\Theta_{11}^{\dagger}\Theta_{12} - \Theta_{22})^{\frac{1}{2}}\Big), \end{cases} \tag{3.61}$$

then both V^+ and V^- are finite and given by (3.58) and (3.60), respectively. Moreover, if

$$\mathcal{R}(\Theta_{12}) \subseteq \overline{\mathcal{R}(\Theta_{11})}, \qquad \mathcal{R}(\Theta_{21}) \subseteq \overline{\mathcal{R}(\Theta_{22})}, \tag{3.62}$$

then the game admits a value.

Proof. We prove (ii) first. Recall that

$$J(u_1, u_2) = \langle\, \Theta_{11}u_1, u_1 \,\rangle + 2\,\langle\, v_1 + \Theta_{12}u_2, u_1 \,\rangle + \langle\, \Theta_{22}u_2, u_2 \,\rangle + 2\,\langle\, v_2, u_2 \,\rangle\,. \tag{3.63}$$

We define

$$J_2(u_2) = \inf_{u_1 \in \mathcal{H}_1} J(u_1, u_2), \tag{3.64}$$

with the domain

$$\mathcal{D}(J_2) \stackrel{\Delta}{=} \{u_2 \in \mathcal{H}_2 \;\big|\; J_2(u_2) > -\infty\}. \tag{3.65}$$

According to Proposition 3.2, $\mathcal{D}(J_2)\neq\phi$ if and only if

$$\Theta_{11}\geq 0,\qquad v_1\in\mathcal{R}(\Theta_{11}^{\frac12})+\mathcal{R}(\Theta_{12}),\tag{3.66}$$

and $\mathcal{D}(J_2)$ is characterized by the following:

$$\mathcal{D}(J_2)=\Big\{u_2\in\mathcal{H}_2\;\big|\;v_1+\Theta_{12}u_2\in\mathcal{R}(\Theta_{11}^{\frac12})\Big\}\equiv\Theta_{12}^{-1}\Big(\mathcal{R}(\Theta_{11}^{\frac12})-v_1\Big).\tag{3.67}$$

Moreover, for any $u_2\in\mathcal{D}(J_2)$, one has

$$J_2(u_2)\equiv\inf_{u_1\in\mathcal{H}_1}J(u_1,u_2)=-|(\Theta_{11}^\dagger)^{\frac12}(v_1+\Theta_{12}u_2)|^2+\langle\,\Theta_{22}u_2,u_2\,\rangle+2\,\langle\,v_2,u_2\,\rangle\,.\tag{3.68}$$

From (3.67), we see that $\mathcal{D}(J_2)$ is the pre-image, under linear operator Θ_{12}, of the linear space $\mathcal{R}(\Theta_{11}^{\frac12})$ translated by a vector v_1. Thus, in general, $\mathcal{D}(J_2)$ is not necessarily a linear space (could even be empty), but it is a convex set. Thus, $J_2(\cdot)$ is a convex functional defined on $\mathcal{D}(J_2)$.

Now, if we assume that $v_1\in\mathcal{R}(\Theta_{11}^{\frac12})$, which is still weaker than the second condition in (3.59), then (3.67) becomes

$$\mathcal{D}(J_2)=\Big\{u_2\in\mathcal{H}_2\;\big|\;\Theta_{12}u_2\in\mathcal{R}(\Theta_{11}^{\frac12})\Big\}\equiv\Theta_{12}^{-1}\Big(\mathcal{R}(\Theta_{11}^{\frac12})\Big),\tag{3.69}$$

which is a linear space (thus, it is always non-empty). Moreover, (3.68) becomes

$$\begin{aligned}J_2(u_2)\;&=-|(\Theta_{11}^\dagger)^{\frac12}v_1+(\Theta_{11}^\dagger)^{\frac12}\Theta_{12}u_2|^2+\langle\,\Theta_{22}u_2,u_2\,\rangle+2\,\langle\,v_2,u_2\,\rangle\\&=\langle\,\Theta_{22}u_2,u_2\,\rangle-|(\Theta_{11}^\dagger)^{\frac12}\Theta_{12}u_2|^2-2\,\langle(\Theta_{11}^\dagger)^{\frac12}v_1,(\Theta_{11}^\dagger)^{\frac12}\Theta_{12}u_2\,\rangle\\&\qquad+2\,\langle\,v_2,u_2\,\rangle-|(\Theta_{11}^\dagger)^{\frac12}v_1|^2.\end{aligned}\tag{3.70}$$

Further, if the second condition in (3.59) (i.e., $v_1\in\mathcal{R}(\Theta_{11})$) holds, then for any $u_2\in\Theta_{12}^{-1}\Big(\mathcal{R}(\Theta_{11})\Big)$, which is dense in $\Theta_{12}^{-1}\Big(\mathcal{R}(\Theta_{11}^{\frac12})\Big)$, one has

$$J_2(u_2)=\langle(\Theta_{22}-\Theta_{21}\Theta_{11}^\dagger\Theta_{12})u_2,u_2\,\rangle+2\,\langle\,v_2-\Theta_{21}\Theta_{11}^\dagger v_1,u_2\,\rangle-|(\Theta_{11}^\dagger)^{\frac12}v_1|^2.\tag{3.71}$$

Note that, in general, $\Theta_{22}-\Theta_{21}\Theta_{11}^\dagger\Theta_{12}$ is an unbounded operator with domain $\Theta_{12}^{-1}\Big(\mathcal{R}(\Theta_{11})\Big)$. Thus, by Proposition 3.2, $J_2(\cdot)$ admits a finite supremum if and only the third and fourth conditions in (3.59) hold, and in this case, V^- is given by (3.60). This proves (ii).

The proof of (i) is similar.

Finally, we prove (iii). Note that (3.61) implies (3.57) and (3.59). Hence, both V^+ and V^- are finite. Consequently, the existence of the value is equivalent to the following:

$$\begin{aligned}&\Big|[(\Theta_{21}\Theta_{11}^\dagger\Theta_{12}-\Theta_{22})^\dagger]^{\frac12}(v_2-\Theta_{21}\Theta_{11}^\dagger v_1)\Big|^2\\&\quad+\Big|[(\Theta_{11}-\Theta_{12}\Theta_{22}^\dagger\Theta_{21})^\dagger]^{\frac12}(v_1-\Theta_{12}\Theta_{22}^\dagger v_2)\Big|^2=\Big|(\Theta_{11}^\dagger)^{\frac12}v_1\Big|^2+\Big|(-\Theta_{22}^\dagger)^{\frac12}v_2\Big|^2.\end{aligned}\tag{3.72}$$

To show the above (under our conditions), we denote

$$\left\{\begin{array}{l}K=\Theta_{11}-\Theta_{12}\Theta_{22}^\dagger\Theta_{21},\quad\Phi=(K^\dagger)^{\frac12},\\ L=\Theta_{21}\Theta_{11}^\dagger\Theta_{12}-\Theta_{22},\quad\Psi=(L^\dagger)^{\frac12},\\ P_1=P_{\overline{\mathcal{R}(\Theta_{11})}},\quad P_2=P_{\overline{\mathcal{R}(\Theta_{22})}},\quad P_K=P_{\overline{\mathcal{R}(K)}},\quad P_L=P_{\overline{\mathcal{R}(L)}}\,.\end{array}\right.\tag{3.73}$$

By (3.61), we know that

$$0 \le \Theta_{11} \le K, \qquad 0 \le -\Theta_{22} \le L. \tag{3.74}$$

Hence,

$$\begin{cases} \mathcal{R}(\Theta_{11}) \subseteq \mathcal{R}(K), \qquad \mathcal{R}(\Theta_{22}) \subseteq \mathcal{R}(L), \\ 0 \le P_1 \le P_K, \qquad 0 \le P_2 \le P_L. \end{cases} \tag{3.75}$$

Note that under (3.62), one has

$$P_1\Theta_{12} = \Theta_{12}, \qquad P_2\Theta_{21} = \Theta_{21}.$$

Hence, the left hand side of (3.72) becomes

$$\begin{aligned} &\Big|\Phi v_1 - \Phi P_1\Theta_{12}\Theta_{22}^\dagger v_2\Big|^2 + \Big|\Psi v_2 - \Psi P_2\Theta_{21}\Theta_{11}^\dagger v_1\Big|^2 \\ &\quad = \left|\begin{pmatrix} \Phi\Theta_{11}^{\frac{1}{2}} & \Phi P_1\Theta_{12}(-\Theta_{22}^\dagger)^{\frac{1}{2}} \\ -\Psi P_2\Theta_{21}(\Theta_{11}^\dagger)^{\frac{1}{2}} & \Psi(-\Theta_{22})^{\frac{1}{2}} \end{pmatrix}\begin{pmatrix} (\Theta_{11}^\dagger)^{\frac{1}{2}} v_1 \\ (-\Theta_{22}^\dagger)^{\frac{1}{2}} v_2 \end{pmatrix}\right|^2 \\ &\quad \equiv \left|\Lambda\begin{pmatrix} (\Theta_{11}^\dagger)^{\frac{1}{2}} v_1 \\ (-\Theta_{22}^\dagger)^{\frac{1}{2}} v_2 \end{pmatrix}\right|^2. \end{aligned}$$

Note that

$$\begin{aligned} \Lambda\Lambda^\top &= \begin{pmatrix} \Phi\Theta_{11}^{\frac{1}{2}} & \Phi P_1\Theta_{12}(-\Theta_{22}^\dagger)^{\frac{1}{2}} \\ -\Psi P_2\Theta_{21}(\Theta_{11}^\dagger)^{\frac{1}{2}} & \Psi(-\Theta_{22})^{\frac{1}{2}} \end{pmatrix} \\ &\quad \cdot \begin{pmatrix} \Theta_{11}^{\frac{1}{2}}\Phi & -(\Theta_{11}^\dagger)^{\frac{1}{2}}\Theta_{12}P_2\Psi \\ (-\Theta_{22}^\dagger)^{\frac{1}{2}}\Theta_{21}P_1\Phi & (-\Theta_{22})^{\frac{1}{2}}\Psi \end{pmatrix} \\ &= \begin{pmatrix} \Phi K\Phi & \Phi P_1[\Theta_{12}P_2 - P_1\Theta_{12}]P_2\Psi \\ \Psi P_2[P_2\Theta_{21} - \Theta_{21}P_1]P_1\Phi & \Psi L\Psi \end{pmatrix} \\ &= \begin{pmatrix} P_K & 0 \\ 0 & P_L \end{pmatrix}. \end{aligned}$$

Therefore, restricted on $\overline{\mathcal{R}(\Theta_{11})} \oplus \overline{\mathcal{R}(\Theta_{22})}$, Λ is a unitary operator. Hence, (3.72) holds, proving the existence of the value. $\square$

We point out that conditions (3.61) and (3.62) do not necessarily imply either (3.33) or (3.35). Note that, similar to Corollary 3.7, the above proposition also follows a leader-follower fashion. It seems that by being a follower, the player has some advantages. In the case that the value exists, roughly speaking, none of the players will have such an advantage by being a follower.

The above proposition does not completely answer the question if the finiteness of both upper and lower values implies the existence of the value. We conjecture that there is a case for which both upper and lower values are finite but they are not equal.

4. **Open-Loop LQ Games.**

In this section we discuss our LQ differential game with both players using open-loop controls. We will mainly apply the results from the previous section, together with some theory of BSDEs and FBSDEs ([22], [19]; see also [24]) to approach the open-loop game.

Let $\Phi(\cdot) \in L^2_{\mathcal{F}}(\Omega; C([0,T]; \mathbb{R}^{n\times n}))$ be the solution of the following SDE:

$$d\Phi(t) = A(t)\Phi(t)dt + C(t)\Phi(t)dW(t), \qquad \Phi(0) = I. \tag{4.1}$$

Then, it is known that for any $x \in \mathbb{R}^n$, and $u(\cdot) \equiv (u_1(\cdot), u_2(\cdot)) \in \mathcal{U}_1 \times \mathcal{U}_2$, the corresponding state process $X(\cdot) \equiv X(\cdot\,; x, u_1(\cdot), u_2(\cdot))$ can be represented by

$$\begin{aligned} X(t) &= \Phi(t)x + \Phi(t)\int_0^t \Phi(\tau)^{-1}\big[B_1(\tau)u_1(\tau) + B_2(\tau)u_2(\tau)\big]d\tau \\ &\quad + \Phi(t)\int_0^t \Phi(\tau)^{-1}\big[D_1(\tau)u_1(\tau) + D_2(\tau)u_2(\tau)\big]dW(\tau) \\ &\equiv \mathcal{A}(t)x + \mathcal{B}_1[u_1(\cdot)](t) + \mathcal{B}_2[u_2(\cdot)](t) \\ &\equiv \mathcal{A}(t)x + \mathcal{B}[u(\cdot)](t), \qquad t \in [0,T]. \end{aligned} \tag{4.2}$$

Here, $\mathcal{B} : L^2_{\mathcal{F}}(0,T;\mathbb{R}^{m_1+m_2}) \to L^2_{\mathcal{F}}(0,T;\mathbb{R}^n)$ is defined by

$$\mathcal{B}[u(\cdot)](t) = X(t; 0, u(\cdot)). \tag{4.3}$$

We also define $\widehat{\mathcal{B}} : L^2_{\mathcal{F}}(0,T;\mathbb{R}^{m_1+m_2}) \to L^2_{\mathcal{F}_T}(\Omega;\mathbb{R}^n)$ by

$$\widehat{\mathcal{B}}[u(\cdot)] = \mathcal{B}[u(\cdot)](T) \equiv X(T; 0, u(\cdot)). \tag{4.4}$$

Clearly, both $\mathcal{B}$ and $\widehat{\mathcal{B}}$ are bounded linear operators, so are their adjoint operators $\mathcal{B}^* : L^2_{\mathcal{F}}(0,T;\mathbb{R}^n) \to L^2_{\mathcal{F}}(0,T;\mathbb{R}^{m_1+m_2})$ and $\widehat{\mathcal{B}}^* : L^2_{\mathcal{F}_T}(\Omega;\mathbb{R}^n) \to L^2_{\mathcal{F}}(0,T;\mathbb{R}^{m_1+m_2})$. Having the above, we are now able to rewrite $J_x(u_1(\cdot), u_2(\cdot))$ as a bilinear form of $(u_1(\cdot), u_2(\cdot))$ explicitly. To this end, we define

$$\begin{cases} \Theta_0 = \mathbb{E}\Big[\displaystyle\int_0^T \mathcal{A}^\top(t)Q(t)\mathcal{A}(t)dt + \mathcal{A}^\top(T)G\mathcal{A}(T)\Big] \\ \Theta_1 = \mathcal{B}^*Q(\cdot)\mathcal{A}(\cdot) + S(\cdot)\mathcal{A}(\cdot) + \widehat{\mathcal{B}}^*G\mathcal{A}(T) \\ \Theta = \mathcal{B}^*Q(\cdot)\mathcal{B} + S(\cdot)\mathcal{B} + \mathcal{B}^*S^\top(\cdot) + R(\cdot) + \widehat{\mathcal{B}}^*G\widehat{\mathcal{B}}. \end{cases} \tag{4.5}$$

Clearly,

$$\begin{cases} \Theta_0 \in \mathcal{S}^n, \\ \Theta_1 : \mathbb{R}^n \to L^2_{\mathcal{F}}(0,T;\mathbb{R}^{m_1+m_2}), \\ \Theta : L^2_{\mathcal{F}}(0,T;\mathbb{R}^{m_1+m_2}) \to L^2_{\mathcal{F}}(0,T;\mathbb{R}^{m_1+m_2}), \end{cases} \tag{4.6}$$

with Θ being self-adjoint. Then we have

$$\begin{aligned} J_x(u_1(\cdot),u_2(\cdot)) &= \mathbb{E}\Big\{\textstyle\int_0^T \Big[\langle Q(t)X(t), X(t)\rangle + 2\langle S(t)X(t), u(t)\rangle \\ &\quad + \langle R(t)u(t), u(t)\rangle\Big]dt + \langle GX(T), X(T)\rangle\Big\} \\ &= \mathbb{E}\Big\{\textstyle\int_0^T \Big[\langle Q(t)\big\{\mathcal{A}(t)x + \mathcal{B}[u(\cdot)](t)\big\}, \mathcal{A}(t)x + \mathcal{B}[u(\cdot)](t)\rangle \\ &\quad + 2\langle S(t)\big\{\mathcal{A}(t)x + \mathcal{B}[u(\cdot)](t)\big\}, u(t)\rangle + \langle R(t)u(t), u(t)\rangle\Big]dt \\ &\quad + \langle G\big\{\mathcal{A}(T)x + \mathcal{B}[u(\cdot)](T)\big\}, \mathcal{A}(T)x + \mathcal{B}[u(\cdot)](T)\rangle\Big\} \\ &\equiv \langle \Theta_0 x, x\rangle + 2\langle \Theta_1 x, u(\cdot)\rangle + \langle \Theta u(\cdot), u(\cdot)\rangle. \end{aligned} \tag{4.7}$$

Hence, our open-loop LQ stochastic differential game problem becomes a game with quadratic cost/payoff functional (4.7) defined on the Hilbert space $\mathcal{U}_1 \times \mathcal{U}_2$. We let

$$\Theta = \begin{pmatrix} \Theta_{11} & \Theta_{12} \\ \Theta_{21} & \Theta_{22} \end{pmatrix}, \qquad \Theta_{ij} : \mathcal{U}_j \to \mathcal{U}_i, \quad i = 1,2. \tag{4.8}$$

Then the following result holds (by Proposition 3.4 from the previous section).

Proposition 4.1. Let (A1)–(A2) hold. For given $x \in \mathbb{R}^n$, the open-loop game admits a saddle point $\hat{u}(\cdot) \equiv (\hat{u}_1(\cdot), \hat{u}_2(\cdot)) \in \mathcal{U}_1 \times \mathcal{U}_2$ if and only if

$$\Theta_{11} \geq 0, \qquad \Theta_{22} \leq 0, \tag{4.9}$$

and

$$\Theta_1 x \in \mathcal{R}(\Theta). \tag{4.10}$$

In this case, any saddle point $\hat{u}(\cdot)$ is a solution of the following equation:

$$\Theta \hat{u} + \Theta_1 x = 0, \tag{4.11}$$

and it admits the following representation:

$$\hat{u}(\cdot) = -\Theta^\dagger \Theta_1 x + [I - \Theta^\dagger \Theta] v(\cdot), \tag{4.12}$$

for some $v(\cdot) \in \mathcal{U}_1 \times \mathcal{U}_2$. The saddle point is unique if and only if $\mathcal{N}(\Theta) = \{0\}$.

Note that when (4.9)–(4.10) hold, for any $v(\cdot) \in \mathcal{U}_1 \times \mathcal{U}_2$, $\hat{u}(\cdot)$ given by (4.12) is a solution of (4.11), and therefore is a saddle point of the game.

We see that (4.9) is equivalent to the convexity of $u_1(\cdot) \mapsto J_0(u_1(\cdot), 0)$ and the concavity of $u_2(\cdot) \mapsto J_0(0, u_2(\cdot))$, and (4.10) is equivalent to the solvability of equation (4.11) for $\hat{u}(\cdot)$. These two conditions seem to be not very explicit. Hence, we would like to look at some other sufficient conditions guaranteeing them. To this end, let us first give a representation for $\mathcal{B}^*$ and $\widehat{\mathcal{B}}^*$. Let us recall that under (A1)–(A2), for any $h(\cdot) \in L^2_{\mathcal{F}}(0,T;\mathbb{R}^n)$ and $\eta \in L^2_{\mathcal{F}_T}(\Omega;\mathbb{R}^n)$, the following linear backward stochastic differential equation (BSDE, for short) admits a unique adapted solution $(Y(\cdot), Z(\cdot))$ ([22], [19]):

$$dY(t) = -\big[A^\top(t)Y(t) + C^\top(t)Z(t) + h(t)\big]dt + Z(t)dW(t), \qquad Y(T) = \eta. \tag{4.13}$$

We have the following result.

Proposition 4.2. Let (A1)–(A2) hold. Then for any $h(\cdot) \in L^2_{\mathcal{F}}(0,T;\mathbb{R}^n)$,

$$\mathcal{B}^*[h(\cdot)](t) = B^\top(t)Y(t) + D^\top(t)Z(t), \qquad t \in [0,T], \tag{4.14}$$

with $(Y(\cdot), Z(\cdot))$ being the adapted solution of (4.13) corresponding to $\eta = 0$ and $h(\cdot)$; and for any $\eta \in L^2_{\mathcal{F}_T}(\Omega;\mathbb{R}^n)$,

$$[\widehat{\mathcal{B}}^*\eta](t) = B^\top(t)Y(t) + D^\top(t)Z(t), \quad t \in [0,T], \tag{4.15}$$

with $(Y(\cdot), Z(\cdot))$ being the adapted solution of (4.13) corresponding to $h(\cdot) = 0$ and η.

Proof. First of all, by [22], BSDE (4.13) admits a unique adapted solution. Next, let $X(\cdot) = X(\cdot\,; 0, u(\cdot)) \equiv \mathcal{B}[u(\cdot)]$. Then using Itô's formula, we have

$$\begin{aligned} d\langle X(t), Y(t)\rangle &= \big\{ \langle A(t)X(t) + B(t)u(t), Y(t)\rangle + \langle X(t), -A^\top(t)Y(t) \\ &\qquad -C^\top(t)Z(t) - h(t)\rangle + \langle C(t)X(t) + D(t)u(t), Z(t)\rangle \big\}dt \\ &\qquad +\big\{ \langle C(t)X(t) + D(t)u(t), Y(t)\rangle + \langle X(t), Z(t)\rangle \big\}dW(t) \\ &= \big[\langle B^\top(t)Y(t) + D^\top(t)Z(t), u(t)\rangle - \langle X(t), h(t)\rangle \big]dt \\ &\qquad +\big\{ \langle C(t)X(t) + D(t)u(t), Y(t)\rangle + \langle X(t), Z(t)\rangle \big\}dW(t). \end{aligned}$$

Consequently,

$$\begin{aligned} &\mathbb{E}\int_0^T \langle B^\top(t)Y(t) + D^\top(t)Z(t), u(t)\rangle\, dt \\ &\quad = \mathbb{E}\Big[\langle X(T), \eta\rangle + \int_0^T \langle X(t), h(t)\rangle\, dt \Big] \\ &\quad = \langle \widehat{\mathcal{B}}u(\cdot), \eta\rangle + \langle \mathcal{B}u(\cdot), h(\cdot)\rangle = \langle u(\cdot), \widehat{\mathcal{B}}^*\eta + \mathcal{B}^*h(\cdot)\rangle. \end{aligned} \tag{4.16}$$

Thus, by the linearity of (4.13), we obtain (4.14) and (4.15) immediately. $\square$

The following result gives an equivalent condition for (4.10) (or the solvability of (4.11)).

Proposition 4.3. Let (A1)–(A2) hold. For given $x \in \mathbb{R}^n$, the following holds:

$$\Theta_1 x + \Theta \hat{u}(\cdot) = S(\cdot)X(\cdot) + B^\top(\cdot)Y(\cdot) + D^\top(\cdot)Z(\cdot) + R(\cdot)\hat{u}(\cdot), \tag{4.17}$$

where $(X(\cdot), Y(\cdot), Z(\cdot))$ is an adapted solution of the following forward-backward stochastic differential equation (FBSDE, for short):

$$\begin{cases} dX = [AX + B\hat{u}]dt + [CX + D\hat{u}]dW, \\ dY = -[QX + A^\top Y + C^\top Z + S^\top \hat{u}]dt + ZdW, \\ X(0) = x, \qquad Y(T) = GX(T). \end{cases} \tag{4.18}$$

Consequently, condition (4.10) holds if and only if there exists a $\hat{u}(\cdot) \in \mathcal{U}_1 \times \mathcal{U}_2$ such that

$$S(t)X(t) + B^\top(t)Y(t) + D^\top(t)Z(t) + R(t)\hat{u}(t) = 0, \qquad t \in [0, T], \text{ a.s.} \tag{4.19}$$

Proof. Let $(X_0(\cdot), Y_0(\cdot), Z_0(\cdot))$ be the adapted solution of (4.18) corresponding to $\hat{u}(\cdot) = 0$ (only depending on x), and let $(\widehat{X}(\cdot), \widehat{Y}(\cdot), \widehat{Z}(\cdot))$ be the adapted solution of (4.18) corresponding to $x = 0$ (only depending on $\hat{u}(\cdot)$). Then by Proposition 4.2, it is straightforward that

$$\Theta_1 x = S(\cdot)X_0(\cdot) + B^\top(\cdot)Y_0(\cdot) + D^\top(\cdot)Z_0(\cdot), \tag{4.20}$$

and

$$\Theta \hat{u}(\cdot) = S(\cdot)\widehat{X}(\cdot) + B^\top(\cdot)\widehat{Y}(\cdot) + D^\top(\cdot)\widehat{Z}(\cdot) + R(\cdot)\hat{u}(\cdot). \tag{4.21}$$

Let us call

$$X(\cdot) = X_0(\cdot) + \widehat{X}(\cdot), \ Y(\cdot) = Y_0(\cdot) + \widehat{Y}(\cdot), \ Z(\cdot) = Z_0(\cdot) + \widehat{Z}(\cdot).$$

Then $(X(\cdot), Y(\cdot), Z(\cdot))$ is an adapted solution of (4.18), and (4.17) holds. Hence, our conclusion follows. □

Note that (4.18)–(4.19) is a necessary condition for $\hat{u}(\cdot)$ to be an open-loop saddle point of the game. Moreover, by Proposition 4.1, we know that if (4.9) holds, then (4.18)–(4.19) is also a sufficient condition for $\hat{u}(\cdot)$ to be an open-loop saddle point of the game.

In a similar nature, we have the following result concerning condition (4.9).

Proposition 4.4. Let (A1)–(A2) hold. For $i = 1, 2$ and any $u_i(\cdot) \in \mathcal{U}_i$, let $(X_i(\cdot), Y_i(\cdot), Z_i(\cdot))$ be the adapted solution of the following:

$$\begin{cases} dX_i = [AX_i + B_i u_i]dt + [CX_i + D_i u_i]dW, \\ dY_i = -[QX_i + A^\top Y_i + C^\top Z_i + S_i^\top u_i]dt + Z_i dW, \\ X_i(0) = 0, \qquad Y_i(T) = GX_i(T). \end{cases} \tag{4.22}$$

Then

$$\begin{cases} \Theta_{11} \geq 0 \iff \langle S_1(\cdot)X_1(\cdot) + B_1^\top(\cdot)Y_1(\cdot) + D_1^\top(\cdot)Z_1(\cdot) + R_{11}(\cdot)u_1(\cdot), u_1(\cdot) \rangle \geq 0, \\ \qquad\qquad\qquad \forall u_1(\cdot) \in \mathcal{U}_1, \\ \Theta_{22} \leq 0 \iff \langle S_2(\cdot)X_2(\cdot) + B_2^\top(\cdot)Y_2(\cdot) + D_2^\top(\cdot)Z_2(\cdot) + R_{22}(\cdot)u_2(\cdot), u_2(\cdot) \rangle \leq 0, \\ \qquad\qquad\qquad \forall u_2(\cdot) \in \mathcal{U}_2. \end{cases} \tag{4.23}$$

Proof. By (4.21), we have

$$\begin{pmatrix} \Theta_{11} \\ \Theta_{21} \end{pmatrix} u_1(\cdot) = S(\cdot)X_1(\cdot) + B^\top(\cdot)Y_1(\cdot) + D^\top(\cdot)Z_1(\cdot) + \begin{pmatrix} R_{11}(\cdot) \\ R_{21}(\cdot) \end{pmatrix} u_1(\cdot).$$

$$\begin{pmatrix} \Theta_{12} \\ \Theta_{22} \end{pmatrix} u_2(\cdot) = S(\cdot)X_2(\cdot) + B^\top(\cdot)Y_2(\cdot) + D^\top(\cdot)Z_2(\cdot) + \begin{pmatrix} R_{12}(\cdot) \\ R_{22}(\cdot) \end{pmatrix} u_2(\cdot).$$

Hence,

$$\langle\, \Theta_{ii}u_i(\cdot), u_i(\cdot)\,\rangle = \langle\, S_i(\cdot)X_i(\cdot)+B_i^\top(\cdot)Y_i(\cdot)+D_i^\top(\cdot)Z_i(\cdot)+R_{ii}(\cdot)u_i(\cdot), u_i(\cdot)\,\rangle, \quad i=1,2.$$

Therefore, (4.23) follows. □

We see that (4.18)–(4.19) gives a coupled FBSDE (the coupling is given through (4.19)). For such an FBSDE, let us look at the solvability via the idea of Four-Step Scheme ([18], [19]). More precisely, let $(X(\cdot), Y(\cdot), Z(\cdot), \hat u(\cdot))$ be $\mathbb{F}$-adapted satisfying (4.18)–(4.19), and suppose that one has the following relation:

$$Y(t) = P(t)X(t), \qquad t \in [0,T], \tag{4.24}$$

where $P(\cdot)$ is an $\mathcal{S}^n$-valued deterministic function. By Itô's formula, we have (suppressing t)

$$\begin{aligned} &-[QX + A^\top Y + C^\top Z + S^\top \hat u]dt + ZdW \\ &\quad = dY = [\dot P X + P(AX + B\hat u)]dt + P(CX + D\hat u)dW \end{aligned} \tag{4.25}$$

Comparing the drift and diffusion terms, we see that one should have (note (4.24))

$$(\dot P + PA + A^\top P + Q)X + (PB + S^\top)\hat u + C^\top Z = 0, \tag{4.26}$$

and

$$Z = PCX + PD\hat u. \tag{4.27}$$

Combining (4.19) and (4.27), we have

$$(B^\top P + D^\top PC + S)X + (D^\top PD + R)\hat u = 0. \tag{4.28}$$

Now suppose the following range condition holds:

$$\mathcal{R}\Big(B^\top(t)P(t) + D^\top(t)P(t)C(t) + S(t)\Big) \subseteq \mathcal{R}\Big(D^\top(t)P(t)D(t) + R(t)\Big), \quad t \in [0,T]. \tag{4.29}$$

Then (4.28) is implied by

$$\hat u = -(D^\top PD + R)^\dagger (B^\top P + D^\top PC + S)X. \tag{4.30}$$

If we take $\hat u$ in such a way, (4.26) becomes (note (4.27))

$$\begin{aligned} 0 &= (\dot P + PA + A^\top P + Q)X + (PB + S^\top)\hat u + C^\top Z \\ &= (\dot P + PA + A^\top P + Q + C^\top PC)X + (PB + C^\top PD + S^\top)\hat u \\ &= \big[\dot P + PA + A^\top P + C^\top PC + Q \\ &\qquad -(PB + C^\top PD + S^\top)(D^\top PD + R)^\dagger (B^\top P + D^\top PC + S)\big]X. \end{aligned} \tag{4.31}$$

Hence, $P(\cdot)$ should satisfies the following Riccati equation:

$$\begin{cases} \dot P + PA + A^\top P + C^\top PC + Q \\ \qquad -(B^\top P + D^\top PC + S)^\top (D^\top PD + R)^\dagger (B^\top P + D^\top PC + S) = 0, \\ P(T) = G, \\ \mathcal{R}(B^\top P + D^\top PC + S) \subseteq \mathcal{R}(D^\top PD + R), \end{cases} \tag{4.32}$$

This proves the following proposition.

Theorem 4.5. Suppose Riccati equation (4.32) admits a solution $P(\cdot)$. Then for any $x \in \mathbb{R}^n$, (4.18)–(4.19) admits an adapted solution $(X(\cdot), Y(\cdot), Z(\cdot), \hat{u}(\cdot))$. Furthermore, if (4.9) holds, then the game admits an open-loop saddle point $\hat{u}(\cdot)$, and it admits a feedback representation (4.30).

Note that (4.30) is just a closed-loop representation of an open-loop saddle point. It does not mean that such a $\hat{u}(\cdot)$ is a closed-loop saddle point.

The above result relies on the solvability of Riccati equation (4.32) (besides condition (4.9)). Thus, one might desire to have some direct solvability result for the linear FBSDE (4.18)–(4.19). To this end, let us assume the following:

$$\begin{cases} \mathcal{R}\Big((S(t), B^\top(t), D^\top(t))\Big) \subseteq \mathcal{R}\Big(R(t)\Big), \qquad t \in [0,T], \\ R^\dagger(\cdot) \in L^\infty(0,T;\mathcal{S}^m). \end{cases} \tag{4.33}$$

Then (4.19) is implied by

$$u(t) = -R^\dagger(t)\big[S(t)X(t) + B^\top(t)Y(t) + D^\top(t)Z(t)\big], \qquad t \in [0,T]. \tag{4.34}$$

Note that in the case that $R^{-1}(\cdot)$ exists and bounded, (4.33) is always true and (4.19) is equivalent to (4.34). Substituting (4.34) into (4.18), we obtain (suppressing t)

$$\begin{cases} dX = \Big[(A - BR^\dagger S)X - BR^\dagger B^\top Y - BR^\dagger D^\top Z\Big]dt \\ \qquad\qquad +\Big[(C - DR^\dagger S)X - DR^\dagger B^\top Y - DR^\dagger D^\top Z\Big]dW, \\ dY = -\Big[(Q - S^\top R^\dagger S)X + (A - BR^\dagger S)^\top Y + (C - DR^\dagger S)^\top Z\Big]dt + ZdW, \\ X(0) = x, \qquad Y(T) = GX(T). \end{cases} \tag{4.35}$$

If the above (coupled) linear FBSDE admits an adapted solution $(X(\cdot), Y(\cdot), Z(\cdot))$, then by defining $u(\cdot)$ through (4.34), we will have (4.19). Now, (4.35) is a linear coupled FBSDE with time-varying deterministic coefficients. Such kind of FBSDEs have been carefully discussed in [26]. Let us recall one result for such an equation. To this end, we introduce the following notation:

$$\begin{cases} \mathcal{A}(t) = \begin{pmatrix} A - BR^\dagger S & -BR^\dagger B^\top \\ -(Q - S^\top R^\dagger S) & -(A - BR^\dagger S)^\top \end{pmatrix}, \quad \mathcal{C}(t) = \begin{pmatrix} -BR^\dagger D^\top \\ (C - DR^\dagger S)^\top \end{pmatrix}, \\ \mathcal{A}_1(t) = \begin{pmatrix} C - DR^\dagger S & -DR^\dagger B^\top \\ 0 & 0 \end{pmatrix}, \quad \mathcal{C}_1(t) = \begin{pmatrix} -DR^\dagger D^\top \\ I \end{pmatrix}. \end{cases} \tag{4.36}$$

Next, let $\Psi(\cdot\,,\cdot)$ be the solution of the following ODE:

$$\frac{d}{dt}\Psi(t,s) = \mathcal{A}(t)\Psi(t,s), \quad t \in [s,T], \qquad \Psi(s,s) = I. \tag{4.37}$$

According to [26], we have the following result.

Proposition 4.6. Let $\mathcal{A}(\cdot), \mathcal{A}_1(\cdot), \mathcal{C}(\cdot)$, and $\mathcal{C}_1(\cdot)$ be all bounded. Suppose

$$\left\{(-G, I)\Psi(T,0)\begin{pmatrix} 0 \\ I \end{pmatrix}\right\}^{-1} \text{ exists,} \tag{4.38}$$

and

$$\left\{(-G, I)\Psi(T,\cdot)\mathcal{C}_1(\cdot)\mathcal{C}_1(\cdot)^T\Psi(T,\cdot)^T\begin{pmatrix} -G^T \\ I \end{pmatrix}\right\}^{-1} \in L^\infty(0,T;\mathcal{S}^n). \tag{4.39}$$

Moreover, suppose

$$\text{either} \quad \mathcal{A}_1(\cdot) = 0, \quad \text{or} \quad \mathcal{C}(\cdot) = 0. \tag{4.40}$$

Then for any $x \in \mathbb{R}^n$, FBSDE (4.35) is solvable.

Note that for our case, conditions in (4.40) are the same as follows:

$$D(t)R^{\dagger}(t)B^{\top}(t) = 0, \quad D(t)R^{\dagger}(t)S(t) = C(t), \qquad t \in [0, T]. \tag{4.41}$$

These are kind of compatibility conditions among some of the coefficient matrices. A nontrivial example for the first condition in (4.41) is that

$$B(t) = D(t) = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \qquad R(t) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad t \in [0, T].$$

One can easily cook up higher dimensional examples.

Finally, according to Corollary 3.5, the existence of an open-loop saddle point for the game is guaranteed by the unique solvability of two LQ problems. Hence, we have the following result.

Proposition 4.7. Let (A1)–(A2) hold. Suppose the following two Riccati equations admit solutions $P_i(\cdot)$, respectively: $(i, j = 1, 2,\ i \neq j)$

$$\begin{cases} \dot{P}_i + P_i A + A^{\top} P_i + C^{\top} P_i C + Q \\ \quad -\big(B_i^{\top} P_i + D_i^{\top} P_i C + S_i\big)^{\top} \big(D_i^{\top} P_i D_i + R_{ii}\big)^{-1} \big(B_i^{\top} P_i + D_i^{\top} P_i C + S_i\big) = 0, \\ P_i(T) = G, \\ (-1)^{i+1} [D_i^{\top} P_i D_i + R_{ii}] \geq \delta I > 0, \end{cases} \tag{4.42}$$

where $\delta > 0$. Then the game admits an open-loop saddle point.

Proof. According to a standard stochastic LQ theory (see [27]), when Riccati equations (4.42) admit solutions $P_i(\cdot)$ $(i = 1, 2)$, one can obtain a unique minimizer for $J_x(u_1(\cdot), 0)$ and a unique maximizer for $J_x(0, u_2(\cdot))$. Hence, Corollary 3.5 applies. $\square$

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