Locating a Recycling Center: The General Density Case

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Abstract: The present paper considers a municipality that has a landfill (fixed in location) and plans to optimally locate a “recycling center” in order to minimize transportation costs. The transportation problem consists of two stages: the first stage is the transportation of waste from the households to the recycling center. Households are distributed (not necessarily uniformly) over the two-dimensional city. The second stage is the transportation of the non-recyclables from the recycling center to the landfill. A precise description of the optimal location of the recycling center is found which depends on the density function, the proportion of waste recycled, and the location of the landfill. JEL: R1.

Recycling has become a part of the waste management plans of many municipalities in the United States in the last ten years. While the exact configuration of these plans vary, transportation is always a major cost component, as indeed it is for waste management in general. The present paper considers a municipality which has a landfill (fixed in location) and plans to optimally locate a “recycling center.” It is supposed that all waste generated by households is transported first to the recycling center, where the recyclables are sorted from the non-recyclables. The recyclables are taken away by a commercial recycler (and thus involve no further transportation costs to the city). The non-recyclables on the other hand, must be transported by the city to the landfill. The households themselves are distributed over the two-dimensional area of the city. The formulation of the problem is “general” in the sense that the function describing the density of household waste is abstract – no special form of the density function has been assumed. This kind of transportation problem has been considered in the literature (see Highfill, McAsey, and Weinstein [1994]), but a uniform density is assumed in that paper. It will be shown that a precise description of the optimal location of the recycling center can be given which depends on the density function, the proportion of waste recycled, and the location of the landfill.

The transportation problem of this paper might usefully be characterized as a “two-stage” problem (although without any connotation of the passage of time since the analysis is static). The first stage is the transportation of waste from the households, distributed over the two-dimensional city, to the recycling center. The second stage is the transportation of the non-recyclables from the recycling center to the landfill. The basic strategy of the paper is to prove a general result (given in the Appendix) and then apply the result to the problem at hand. Specifically, the model is given in Section 1 and the optimal location of the recycling center characterized in Section 2.

The present paper extends the literature on optimal location which assumes “two-stage” transportation problems. One recent paper of this kind is Braid [1996]. We generalize one of his cases both by using a general density function and extending the analysis to two dimensions. Weslowsky and Love [1971] consider a transshipment problem in which the transportation proceeds from fixed locations to a discrete number of destinations which are either areas or points. Within each area destination the distributions are uniform. Transportation in our problem runs from areas to the fixed location, and thus runs Westlowsky and Love's problem “backwards,” except that again, we have a general density function. Another difference is that our model permits endpoint solutions while such solutions do not arise in their formulation. The specific location model being generalized in this paper is drawn from Highfill, McAsey, and Weinstein [1994]. The present paper takes its model directly from that paper except that the distribution function is general rather than uniform, and there are no restrictions whatsoever on the (exogenous) landfill location. Highfill, McAsey, and Mou [1997] also generalizes Highfill, McAsey, and Weinstein [1994] by considering two recycling centers rather than one, but in order to do so they reduce the city to a one-dimensional city. Although not as directly relevant to the present paper, there is a growing literature on municipal recycling. While much of it is empirical, there is some theoretical work. For example, Morris and Holthausen [1994] and Kinnaman and Fullerton [1995] investigate the effect of fee structures on household recycling while Highfill and McAsey [1997] conduct a theoretical investigation of the dynamics of landfill exhaustion and recycling.

1. The Basic Model

Consider a city (or county, township, $\ldots$) that wishes to establish a recycling center at a location which minimizes the total transportation cost to the city of hauling residential waste. Assume that the waste generated by the households is distributed spatially according to a (non-negative, bounded, measurable) density function[1] $\rho(x,y) : \mathbb{R}^2 \to \mathbb{R}$. Without loss of generality, we assume that the total amount of waste is 1:

$$\int\int_{\mathbb{R}^2} \rho(x,y)\,dx\,dy = 1$$

and also assume that $\rho(x,y) > 0$ only on some bounded subset of the plane.

Let the fixed coordinates of the landfill be (x_L, y_L). The city wishes to choose coordinates (x_R, y_R) for the recycling center to minimize transportation costs. We use the "Manhattan metric" so that the distance between any two arbitrary points (x_1, y_1) and (x_2, y_2) is assumed to be $|x_1 - x_2| + |y_1 - y_2|$. Assume that the waste is disposed of in two stages. First, all waste is taken to the recycling center where the waste is sorted into recyclables and nonrecyclables. Denote the proportion of waste recycled by γ. (Once sorted, the recyclable waste is no longer of concern to the city because the city has contracted with a commercial recycler for its removal.)[2] Second, the remaining proportion of waste, $1 - \gamma$, is transported to the landfill.

The total transportation cost is

$$\begin{aligned} F(x_R, y_R) = & \int\int_{\mathbb{R}^2} \rho(x,y)(|x - x_R| + |y - y_R|)\,dx\,dy \\ & + (1-\gamma)(|x_R - x_L| + |y_R - y_L|) \end{aligned} \tag{1}$$

The "stage 1" transportation cost, i.e., the cost of transporting the waste between the households and the recycling center, is

$$F_1(x_R, y_R) = \int\int_{\mathbb{R}^2} \rho(x,y)(|x - x_R| + |y - y_R|)\, dx\, dy.$$

The other term of (1) is the "stage 2" costs: the cost of transporting the non-recyclables between the recycling center and the landfill[3]

$$F_2(x_R, y_R) = (1-\gamma)(|x_R - x_L| + |y_R - y_L|).$$

2. The Optimal Location of the Recycling Center

The optimal location of the recycling center is the one that minimizes the sum of the stage 1 (households to recycling center) and stage 2 (recycling center to landfill) transportation costs. Finding such a location is considerably simplified because the stage 1 costs depend on the waste density and the recycling center location but are independent of the landfill location. The stage 2 costs depend on the locations of the landfill and recycling center and the proportion of the waste recycled, but not on the density of waste. The theorem giving the optimal location is found in the Appendix and described next.

The irregularly shaped region in Figure 1 represents a map of the (boundary of the) city. Let K be the rectangle in the figure (whose construction will be described shortly). The basic procedure to find the optimal location of the recycling center is: i) if the landfill is located outside of K, then the optimal location of the recycling center is at the nearest point of K; ii) if the landfill is located inside K, then it is optimal to locate the recycling center at the landfill.

2.A. Reduction to One Dimension

In this sub-section we show how to find the optimal coordinates of the recycling center. Because of the structure of the transportation function, we can reduce the two-dimensional problem to a one-dimensional problem and concentrate on the calculation of the optimal x-coordinate of the recycling center. To this end integrate the y-coordinate out of the density function and define $g(x) = \int_{-\infty}^{\infty} \rho(x,y)\, dy$. Next, define analogs of the stage 1 and 2 costs:

$$G_1(x_R) = \int_{-\infty}^{\infty} g(x)|x - x_R|\, dx \quad \text{and} \quad G_2(x_R) = (1-\gamma)|x_R - x_L| \tag{2}$$

Let $G(x_R) = G_1(x_R) + G_2(x_R)$.

These cost functions are shown in Figure 2. The horizontal axis shows the east-west dimension of the city while the vertical axis shows the total stage 1 costs and the total stage 2 costs. As drawn, the figure assumes that the x-median is to the left of the location of the landfill. Consider the stage 1 costs, $G_1(x_R)$, first. Since g is supported on a bounded set, it follows that $|G_1(x_R)| \to \infty$ as $|x_R| \to \infty$. Thus, since G_1 is continuous, it has a minimum. It is not hard to show that the minimum occurs at the x-median, which we assume for the discussion in this section to be uniquely determined.[4] As seen from Figure 2, locating the recycling center at the

x-median of the waste density would minimize the cost of transporting waste from the households to the recycling center, i.e., $G_1(x_R)$. This would not, of course, minimize the stage 2 costs, as will be explained shortly.

Now consider the stage 2 costs, $G_2(x_R)$. Since G_2 is an absolute value function, its slope is either $(1-\gamma)$ or $(\gamma - 1)$. The minimum of the function is at x_L, the x-coordinate of the landfill. Although clear from the form of G_2, this is also explained by noting that if the recycling center were located at the landfill then the cost of transporting the non-recycled waste between these two locations would be zero. But, not surprisingly, locating the recycling center to minimize the stage 2 costs does not minimize the stage 1 costs.

Consider now the sum of the stage 1 costs and the stage 2 costs. The preceding two paragraphs show that the recycling center must be located between the x-median and the landfill. An instructive method of optimally locating the recycling center is to consider graphically the costs for all possible locations between the median and the landfill. Begin by considering a recycling center located to the right of the median, but near it in Figure 2. In this case, stage 1 costs are low but stage 2 costs are high, and such a location is not likely to be optimal. Shifting the location to the right will increase the stage 1 costs, but decrease the stage 2 costs. Although the terminology is somewhat arbitrary, we will call the marginal reduction of stage 2 costs the "marginal benefit" of such a right-ward shift and the increase in stage 1 costs the "marginal cost" of such a shift. Shifting nearly to the right end of the interval between the median and the landfill will give low stage 2 costs but high stage 1 costs and again such a location is unlikely to be optimal.

The optimal location of the recycling center is found from the graph by beginning a search for such an optimal location at the median and then shifting it to the right until the marginal benefit of such a right-ward shift is equal to the marginal cost of such a shift. The marginal benefit of such a right-ward shift is $1-\gamma$ (the absolute value of the slope of the stage 2 function). The marginal cost is the slope of the stage 1 cost function. Taking the derivative of G_1, the first term of (2), and using the fact that $\int_{-\infty}^{\infty} g(x)\,dx = 1$, this marginal cost is $1 - 2\int_{x_R}^{\infty} g(x)\,dx$. Note that $\int_{x_R}^{\infty} g(x)\,dx$ gives the total amount of waste in the city to the right of the point x_R. The final step of the calculation is to equate the marginal cost and benefit, $1-\gamma = 1 - 2\int_{x_R}^{\infty} g(x)\,dx$, and solve for x_R, the x-coordinate of the optimal location of the recycling center. That is, x_R is chosen in such a way that

$$\int_{x_R}^{\infty} g(x)dx = \frac{\gamma}{2}. \tag{3}$$

In Figure 2, the point x_R is where the slope of the stage 1 cost function is the same (except for sign) as the slope of the stage 2 cost function.[5] This point is, as shown, $x = x''$.

In sum, the x-coordinate of the transportation cost $G(x_R)$ has a critical point, i.e. $G'(x_R) = 0$, where the amount of waste to the right of the coordinate is equal to half of the recycling proportion, γ. For example, if 25% of waste is recycled then x_R is chosen so that 12.5% of the total waste is contained in the area of the city to the right of x_R. As Figure 2 is drawn, this critical point is between the landfill and the x-median, and gives the optimal location for the recycling center.

Figure 3 shows the case when the critical point x'' is not between the landfill and the x-median, while maintaining the assumption that the landfill is to the right of the median. Since the critical value x'' does not lie between the landfill and the x-median, and, as argued above, the optimal location of the recycling center must be between them, it follows that the optimal location must be an endpoint of the interval between the median and the landfill. In this case the optimum is always obtained by locating the recycling center at the landfill, where there are no stage 2 costs.[6]

2.B. The Two-Dimensional Results

The case analyzed in the previous section, where the landfill was to the right of the x-median, was of course only one possibility. We could have just as easily considered the case where the landfill was to the left of the median, or either of the cases on the north-south dimension of the city (i.e., where the landfill is above or below the y-median). The resulting analysis is summarized in a single map of the city, shown in Figure 1. Recall that the axes are the dimensions of the city, and its geographical boundaries are shown. As drawn in Figure 1, we are considering a city whose household waste density is skewed to the northwest giving a median as indicated. Consider now the critical point $x = x''$ found in Figures 2 and 3. Recall that x'' is the point such that the amount of waste to the right of it is equal to half of the recycling proportion, γ. Rewriting condition (3) to reintroduce the original density function ρ, x'' is defined by $\int\int_{x \geq x''} \rho(x,y)\,dx\,dy = \frac{\gamma}{2}$. The other critical points (i.e., when the landfill is to the left of the median on the x axis, and both critical points on the y axis) are constructed similarly. Formally define the critical points x', x'', y', y'' by the following:[7]

$$\int\int_{x \leq x'} \rho(x,y)\,dx\,dy = \frac{\gamma}{2}, \quad \int\int_{x \geq x''} \rho(x,y)\,dx\,dy = \frac{\gamma}{2}$$

and

$$\int\int_{y \leq y'} \rho(x,y)\,dx\,dy = \frac{\gamma}{2}, \quad \int\int_{y \geq y''} \rho(x,y)\,dx\,dy = \frac{\gamma}{2} \ .$$

In general, there are four possible combinations of critical points: (x', y'), (x', y''), (x'', y') and (x'', y''). Denote the rectangle which connects these four points by K. As shown in the Appendix, this rectangle can be used to find the optimal location of the recycling center. The basic procedure to find the optimal location of the recycling center is as follows. A fixed level of recycling γ will determine the critical points x', x'', y', y'' described above and hence the associated rectangle K. If the landfill is located outside of K, the optimal location of the recycling center is at the nearest point of K. If the landfill is located inside K, then it is optimal to locate the recycling center at the landfill. To expand on this procedure, it will be useful to distinguish three cases:

Case 1. Critical point solutions on both axes. If the coordinate of the landfill does not lie between the coordinate of the median and the critical point on either axis, (as with the x-coordinate on Figure 2) then the optimal location of the recycling center will be at one of the four possible combinations of critical points, that is, on a corner of the rectangle K. For example, suppose the landfill is located at point T (above and to the right of the rectangle) in Figure 1. Then the landfill is both to the right of the x-axis critical point and above the y-axis critical point. So the optimal location for the recycling center is at the corner (x'', y'').

Case 2. Recycling Center Coordinate = Landfill Coordinate on both axes. Suppose now that the landfill coordinate is indeed between median and the critical point on both axes (as illustrated in Figure 3 for the x-coordinate). In this case it is optimal to locate the recycling center at the landfill. This case is illustrated in Figure 1 by the landfill R.

Case 3. Mixed Solution. Suppose next that the landfill is located at the point S in Figure 1. Then on the x-axis, the critical point (x'') is optimal (as in Case 1) while on the y-axis, the optimal y-coordinate of the recycling center is the same as the y-coordinate of the landfill (as in Case 2). So in this case, the recycling center will be located on the boundary of the rectangle K, but not at a corner.

3. Conclusion

In the context of this model, different cities are characterized by different landfill locations and different levels of recycling. The effect of the landfill location on the optimal location of the recycling center has been explained via Figure 1. The comparative statics of γ, the proportion of waste recycled, can also be explained using the same figure. The effect in Figure 1 of an increase in γ (i.e., more recycling is accomplished) is that the rectangle K shrinks around the density center of the city. In the extreme case where all waste is recycled ($\gamma = 1$), the rectangle K degenerates to a single point at the center of mass of the city, the location of the landfill is irrelevant, and the optimal strategy is simply to minimize stage 1 costs and locate the recycling center at the density center. If, on the other hand, the proportion of waste recycled decreases, the rectangle K of Figure 1 becomes larger; at the limit it would be the largest possible rectangle whose sides are tangent to the city. In that case, when $\gamma = 0$, the recycling center (now only a "pass through" facility) would either be located at the landfill, if the landfill is in the city, or it would be located on this "largest" rectangle nearest the landfill.

For many cities the problem of finding the optimal location of a recycling center is more complicated than the decision analyzed in this paper. At a minimum other factors such as cite acquisition costs, externalities, and the NIBY phenomenon would have to be considered. On the other hand, the advantage of a simple model is that it can be applied to other transportation problems. Thus the present model could be adapted for such problems as finding the optimal location of a branch library (e.g. Braid [1996]) or of a "break bulk" facility.

NOTES

1. The density function is often proportional to the population density. Additionally, the function ρ need only represent the waste that must be picked up by the city. For example, if the city requires business to contract separately for their own waste disposal, then $\rho(x, y)$ is zero at such locations.

2. Assume the city contracts with a private firm to take away the recyclables for a flat fee, so there are no further variable costs for the recyclables. In all of the analysis of this paper we assume that the transportation costs can be considered independently of any other costs. For this section, we also assume that the per unit transportation costs between the households and the recycling center and between the recycling center and the landfill are the same.

3. Note that the transportation cost in both stages is (proportional to) the product of an amount of waste and a distance (e.g. tons · miles). This is clear in stage 1 where ρ is, say, tons of waste per square mile. In stage 2, there is a "hidden" total amount of waste since we have assumed that the total waste is 1. So too in this case, the units of F_2 are, for example, tons · miles.

4. The x-median is the number x_m so that $\int_{-\infty}^{x_m} g(x)\,dx = \frac{1}{2} = \int_{x_m}^{\infty} g(x)\,dx$. This number may not be uniquely defined if $g(x) = 0$ on a suitable interval. Thus if the cumulative distribution $G(x) = \int_{-\infty}^{x} g$ assumes the constant value $\frac{1}{2}$ on some interval the x-median is not uniquely defined.

5. The intersection of G_1 and G_2 has no significance in this analysis.

6. The theorem of the Appendix shows that in a choice between locating the recycling center at the x-median or the landfill (with the critical point as in Figure 3), transportation costs are minimized by locating the recycling center at the landfill (where stage 2 costs are eliminated) rather than at the x-median where stage 1 costs are minimized.

7. The quantities x', x'', y', y'' might not be uniquely defined by these conditions in case $\rho(x, y) = 0$ on some strip. This will be discussed further in the proof of the theorem.

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APPENDIX

The theorem describing the minimum of the function $F(x_R, y_R)$ from equation (1) is stated in a slightly more general context after first proving the following Lemma. The lemma shows how to solve a one-dimensional version of the original minimization problem. For this, assume that a and b are positive constants and z_L is fixed. Assume that $g(z)$ is a bounded, measurable, non-negative function on $\mathbb{R}$ with support contained in an interval of finite length and satisfying $\int_{-\infty}^{\infty} g(z)\,dz = 1$.

Lemma. *There exists numbers z' and z'' with $z' < z''$ so that the minimum of the function*

$$G(r) = a\int_{-\infty}^{\infty} g(z)|z-r|\,dz + b|r - z_L|$$

occurs at the point of the interval $[z', z'']$ nearest to z_L.

Proof. It is easy to check that G is continuous on $\mathbb{R}$ and continuously differentiable there except at $z = z_L$, but that $z = z_L$ is always a critical point of G. Observe that for any z_L, $G(r) \to \infty$ as $|r| \to \infty$ so that G has a minimum and attains it at a critical point. For $r \neq z_L$ we can differentiate G to get

$$\begin{aligned} G'(r) &= a\int_{-\infty}^{\infty} g(z)\,\mathrm{sgn}(r-z)\,dz + b\,\mathrm{sgn}(r - z_L) \\ &= a\int_{-\infty}^{r} g(z)\,dz - a\int_{r}^{\infty} g(z)\,dz \; + b\,\mathrm{sgn}(r - z_L). \end{aligned}$$

Since $\int_{-\infty}^{-\infty} g(z)\,dz = 1$, we can rewrite G' as

$$G'(r) = \begin{cases} 2a\int_{-\infty}^{r} g(z)\,dz - a + b & \text{if } z_L < r \\ a - 2a\int_{r}^{\infty} g(z)\,dz - b & \text{if } z_L > r \end{cases}.$$

Now assume temporarily that $a > b$ and that there are numbers z' and z'' that are uniquely defined by the conditions

$$\int_{z'}^{\infty} g(z)\,dz = \frac{a-b}{2a} \quad \text{and} \quad \int_{-\infty}^{z''} g(z)\,dz = \frac{a-b}{2a}. \tag{4}$$

The only two possible solutions for $G'(r) = 0$ are the two numbers z' and z''. So there are at most three critical points for G: z', z'', and z_L. The actual minimizer depends on the relative locations of these three numbers. Since $z' < z''$ there are three possible cases to consider.

Case 1. $z_L < z'$. There exist critical points at z_L and z' and $G'(z') = 0$. We assert that $G'(r) < 0$ for $z_L < r < z'$. To see this note that for such r, $\int_{-\infty}^{r} g(z)\,dz < \frac{a-b}{2a}$ so that

$$G'(r) = 2a\int_{-\infty}^{r} g(z)\,dz - a + b < 2a\frac{a-b}{2a} - a + b = 0.$$

So G is decreasing between z_L and z' and thus in the choice between the two critical points, the minimum occurs at z'.

Case 2. $z_L > z''$. The argument is similar to the preceding case.

Case 3. $z' < z_L < z''$. We assert that the equation $G'(r) = 0$ has no solution. In this case we have $\int_r^\infty g(z)dz > \frac{a-b}{2a}$ and $\int_{-\infty}^r g(z)\,dz > \frac{a-b}{2a}$. So for $r < z_L(< z'')$ we have

$$G'(r) = a - 2a\int_r^\infty g(z)dz - b < a - 2a\frac{a-b}{2a} - b = 0$$

and for $z' < z_L < r < z''$ the derivative is

$$G'(r) = 2a\int_{-\infty}^r g(z)\,dz - a + b > 2a\frac{a-b}{2a} - a + b = 0.$$

So the minimum occurs at the only critical point, namely z_L.

This completes the proof in case z' and z'' are uniquely determined (as usual) and $a > b$. In case $a < b$, it is easy to show that $G'(r)$ has a unique sign change at $r = z_L$ so the minimum always occurs at z_L. To finish the proof, we need to consider the case that one or both of z', z'' is not uniquely determined by (4). This occurs in case the density function $g(z)$ is zero on an interval of positive length so that, for example,

$$\int_{-\infty}^{z'} g(z)\,dz = \int_{-\infty}^{z'+\epsilon} g(z)\,dz$$

for some z' and some $\epsilon > 0$.

In case z' is not unique, we define z' (resp. z^*) to be the smallest (resp. largest) value of z satisfying $\int_{-\infty}^{z'} g(z)\,dz = \frac{a-b}{2a}$ so that the integral $\int_{-\infty}^r g(z)\,dz$ is constant on the interval $z' \le r \le z^*$. Similarly if z'' is not uniquely defined by (4), define z'' (resp. z^{**}) to be the largest (resp smallest) value of z satisfying $\int_{z''}^\infty g(z)\,dz = \frac{a-b}{2a}$. (Note that since the support of the density function is bounded, these quantities are finite.)

Consider again Case 1 $(z_L < z')$, but with the alternative assumption $z_L < z^*$. The function $G(r)$ is constant on the interval $[z', z^*]$. To see this note that since $\int_{z'}^{z^*} g(z)\,dz = 0$ we have

$$G(r) = a\int_{-\infty}^{z'} g(z)(r-z)\,dz + a\int_{z^*}^\infty g(z)(z-r)\,dz + b(r - z_L).$$

And so

$$G'(r) = a\int_{-\infty}^{z'} g(z)\,dz - a\int_{z^*}^\infty g(z)\,dz + b = a\frac{a-b}{2a} - a\left(1 - \frac{a-b}{2a}\right) + b = 0.$$

Thus the essential ideas of the proof of Case 1 remain the same: G is decreasing for $r < z'$ and there are no critical points for G with $r > z^*$. Thus G is minimized at any point of the interval $[z', z^*]$.

The argument in Case 2 is similar with z'' replaced by z^{**}. In Case 3, replace the interval $[z', z'']$ with $[z^*, z^{**}]$. Then observe that for $z^* < z_L < z^{**}$, the function $G(r)$ is decreasing for $r < z_L$ and increasing for $r > z_L$ so, as before, there are no critical points other than $r = z_L$ and the minimum occurs there.

Finally, in the event that z_L falls in one of the intervals $[z', z^*]$ or $[z'', z^{**}]$, it is easy to see that $r = z_L$ is the only critical point of $G(r)$ and that the sign of G' changes at the critical point showing that the minimum occurs at $r = z_L$.☐

Assume now, as in the text, that $\rho(x, y)$ is a non-negative, bounded, measurable function having support on a bounded subset of $\mathbb{R}^2$. The goal is to find x_R, y_R to minimize the function

$$\begin{aligned} F(x_R, y_R) = a \int\int_{\mathbb{R}^2} \rho(x, y)(|x - x_R| + |y - y_R|)\, dx\, dy \\ + b\,(|x_R - x_L| + |y_R - y_L|). \end{aligned}$$

Split the F into a sum of two functions

$$\begin{aligned} F(x_R, y_R) = &\left(a \int\int_{\mathbb{R}^2} \rho(x, y)|x - x_R|\, dx\, dy + b|x_R - x_L| \right) \\ &+ \left(a \int\int_{\mathbb{R}^2} \rho(x, y)|y - y_R|\, dx\, dy + b|y_R - y_L| \right). \end{aligned}$$

The first term is a function of x_R and the second a function of y_R. Now apply the lemma to each of these pieces using $g(x) = \int_{-\infty}^{\infty} \rho(x, y)\, dy$ for the first piece and $g(y) = \int_{-\infty}^{\infty} \rho(x, y)\, dx$ for the second. The result is four numbers x', x'', y', and y''. These can be used as x and y coordinates to form the corners of a rectangle K: (x', y'), (x', y''), (x'', y'), (x'', y''). Using this notation the following theorem can now be stated. The proof is, as just described, two applications of the Lemma.

Theorem. *For fixed (x_L, y_L), the function $F(x_R, y_R)$ is minimized by choosing (x_R, y_R) to be the nearest point in the rectangle K to (x_L, y_L). In particular, if $(x_L, y_L) \in K$, then (x_R, y_R) = (x_L, y_L).*

The final point to notice about the theorem is that the rectangle K, although well defined, is arbitrarily chosen in case the (analogs of) conditions (4) do not uniquely describe the points x', x'', y', y''. This will occur in case the density function $\rho(x, y)$ is zero on some strip and so a component of the transportation function is constant in some region. The result is that at least one side of the rectangle K could just as easily have been defined differently. (The best way to describe this is to imagine a rectangle whose boundary is drawn with "thick" lines.) The resulting location of (x_R, y_R) might change, but the total transportation cost would remain the same.

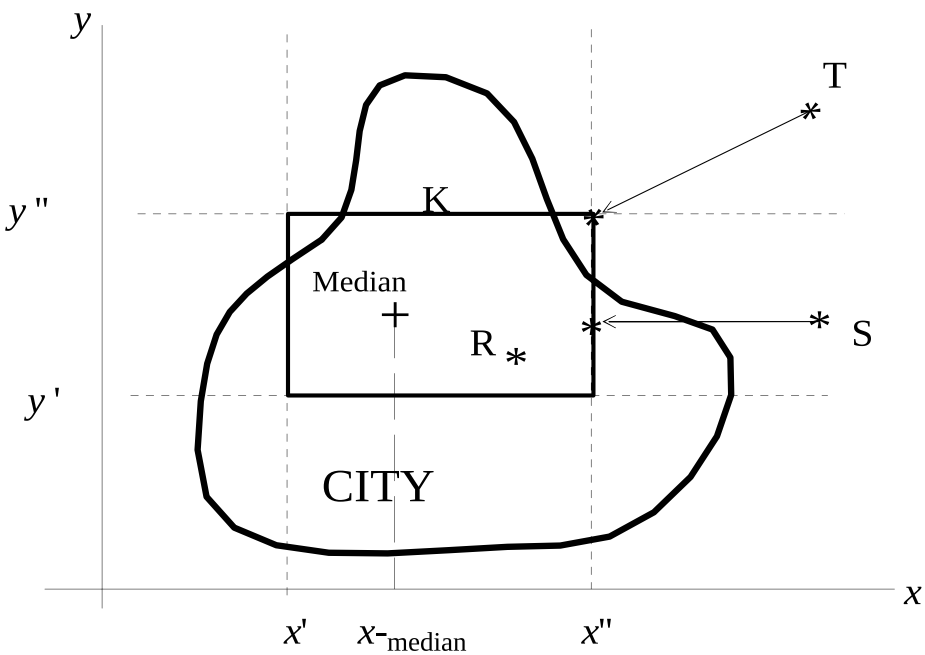

FIGURE 1
The City with Several Possible Landfill Locations

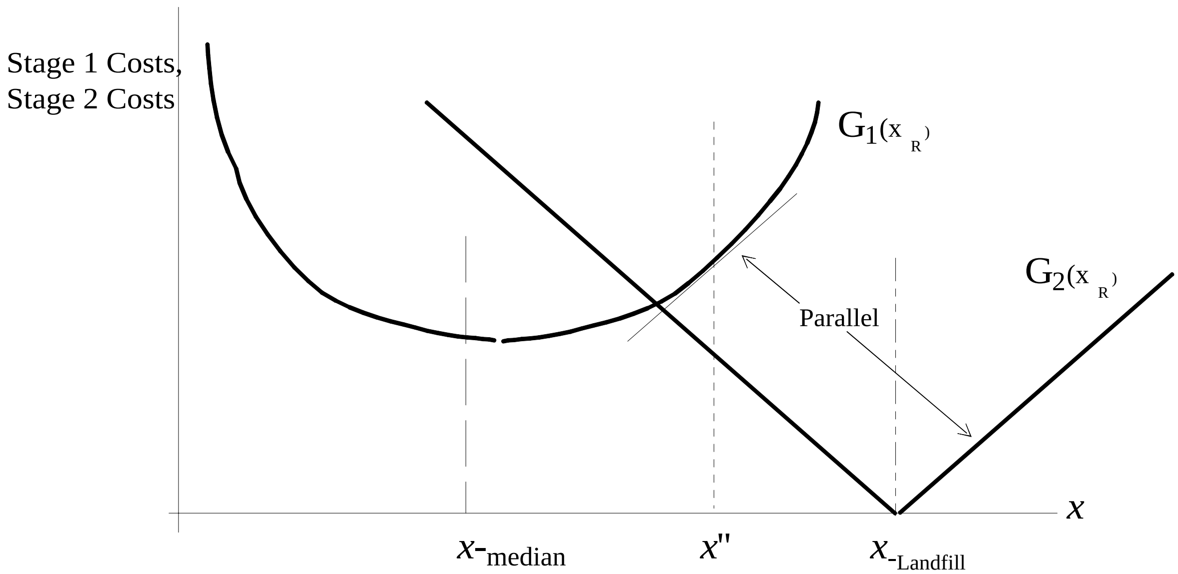

FIGURE 2
Decomposition of Transportation Costs: Critical Point Case

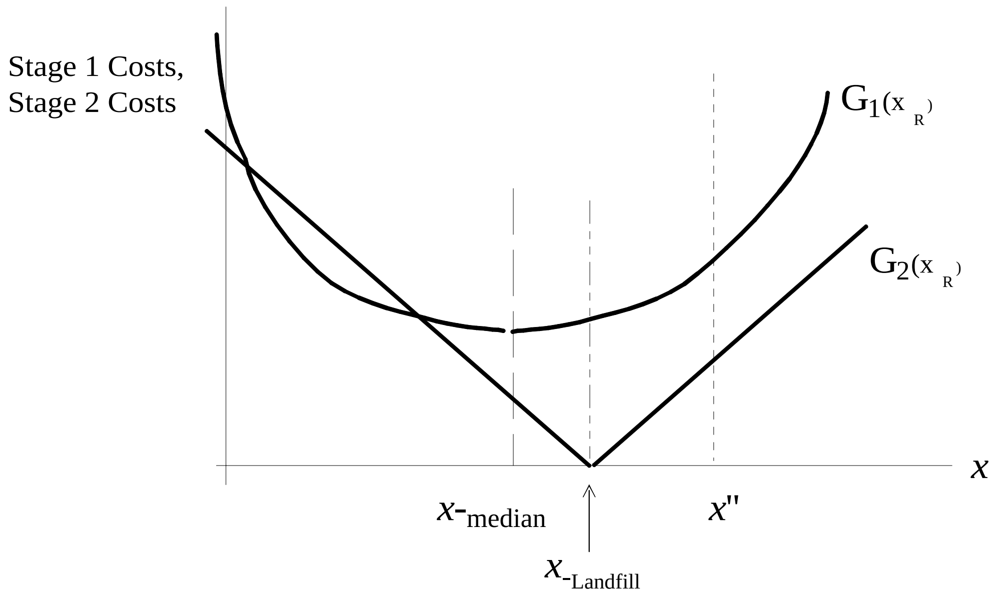

FIGURE 3
Decomposition of Transportation Costs:
Recycling Center at Landfill