Estimation of Maximal Existence Intervals for Solutions to a Riccati Equation via an Upper-Lower Solution Method

Libin Mou*[1] and Stanley R. Liberty**

*Department of Mathematics, Bradley University, Peoria, IL 61625, mou@bradley.edu
**Office of the Provost, Bradley University, Peoria, IL 61625, sliberty@bradley.edu

The authors and colleagues, in work to be submitted for publication, have applied the method of upper and lower solutions to the study of Riccati equations of various types, including those in [1] [2] [3] [4] and [5]. In this note, we show how to estimate the maximal existence interval of the solution to the classical Riccati differential equation.

Denote by $\mathbb{S}^n$ the set of all real symmetric $n \times n$ matrices. We use $G \geq H \ (G > H)$ to denote that $G - H$ is positive semidefinite (definite). For $I = [t_0, t_f]$ or $(-\infty, t_f]$ and $X = R^{n\times n}$ or $\mathbb{S}^n$, denote by $L^\infty(I, X)$ the space of all bounded and measurable $n \times n$ functions from I to X and by $L^{1,\infty}(I, X)$ the space of all $P \in L^\infty(I, X)$ with $P' \in L^\infty(I, X)$. Suppose $A \in L^\infty(I, R^{n\times n})$, $B, Q \in L^\infty(I, \mathbb{S}^n)$ and $P_f \in \mathbb{S}^n$. Consider the classical differential Riccati equation

$$\mathcal{E}(P) \equiv P' + A^T P + PA + Q - PBP = 0, P(t_f) = P_f, \tag{1}$$

Definition. Suppose $P \in L^{1,\infty}(I; \mathbb{S}^n)$. We say that (i) P is an *upper solution* to (1) in I if $\mathcal{E}(P) \leq 0$ and $P(t_f) \geq P_f$, (ii) P is a lower *solution* to (1) in I if $\mathcal{E}(P) \geq 0$ and $P(t_f) \leq P_f$. P is a *solution* if it is both an upper solution and a lower solution.

Many results for equation (1) were proved for the so-called *normal case*; that is, Q, $P_f \geq 0$, which is equivalent to that 0 is a lower solution to (1). All of these results can be generalized for equation (1) under the much weaker condition that (1) has a lower solution P_2. In particular, Q and P_f could be indefinite. Indeed, if P_2 is a lower solution to (1), then $P_2(t_f) \leq P_f$ and $Q_2 \equiv \mathcal{E}(P_2) \geq 0$. Let $K = P - P_2$. Then (1) is equivalent to

$$(A - BP_2)^T K + K(A - BP_2) + Q_2 - KBK = 0,\ K(t_f) = P_f - P_2(t_f),$$

which become *normal* because $Q_2 \geq 0$ and $K(t_f) \geq 0$.

A fundamental property is the following upper-lower solution theorem.

Theorem. *Suppose that (P_1, P_2) is a pair of upper-lower solutions to (1) in I. Then*
(i) $P_1 \geq P_2$ *in* I. (ii) *Equation (1) has a unique solution P with $P_1 \geq P \geq P_2$ in I.*

Consequently, equation (1) has a solution if and only if it has a pair of upper-lower solutions. For example, in the classical case B, Q, $P_f \geq 0$, 0 is a lower solution to (1) while the solution P_1 to the linear equation $P' + A^T P + PA + Q = 0, P(t_f) = P_f$, which always

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exists, is an upper solution to (1). Therefore, (1) must have a solution in this case. The upper-lower solution Theorem also implies other existence results.

Next we assume that A, B and Q are defined on $I = (-\infty, t_f]$. We will use the upper-lower solution Theorem to estimate $I_m(\mathcal{E})$, which denotes the maximal existence interval of the solution to (1). Since $B \in \mathbb{S}^n$, we can write $B = B_p - B_n$, where $B_p, B_n \geq 0$ with $B_pB_n = 0$ are called the positive and negative parts of B. Similarly we write $Q = Q_p - Q_n$ and $P_f = P_{fp} - P_{fn}$. Consider the following equations.

$$\begin{cases} \mathcal{E}_1(P) = P' + A^T P + PA + Q_p + PB_nP = 0, P(t_f) = P_{fp} \\ \mathcal{E}_2(P) = P' + A^T P + PA - Q_n - PB_pP = 0, P(t_f) = -P_{fn} \end{cases} \tag{2}$$

Suppose P_1 and P_2 are the solutions to $\mathcal{E}_1(P) = 0$ and $\mathcal{E}_2(P) = 0$ in their maximal intervals $I_m(\mathcal{E}_1)$ and $I_m(\mathcal{E}_2)$, respectively. It is easily seen that $\mathcal{E}(P_1) \leq \mathcal{E}_1(P_1) \leq 0$ and $\mathcal{E}(P_2) \geq \mathcal{E}_2(P_2) \geq 0$. In other words, (P_1, P_2) is a pair of upper-lower solutions to (1). By the upper-lower solution Theorem, the solution to (1) exists at least in $I_m(\mathcal{E}_1) \cap I_m(\mathcal{E}_2)$, which implies that $I_m(\mathcal{E}) \supset I_m(\mathcal{E}_1) \cap I_m(\mathcal{E}_2)$.

To construct simpler upper and lower solutions, consider the scalar equations.

$$\begin{cases} e_1(p) \equiv p' + \alpha_1 p^2 + \beta_1 p + \gamma_1 = 0,\ p(t_f) = p_{1f}, \\ e_2(p) \equiv p' + \alpha_2 p^2 + \beta_2 p + \gamma_2 = 0, p(t_f) = p_{2f}, \end{cases} \tag{3}$$

where $p_{1f} = \Lambda(P_{fp})$, $\alpha_1 = \Lambda(B_n)$, $\beta_1 = \Lambda(A + A^T)$, $\gamma_1 = \Lambda(Q_p)$, $p_{2f} = \lambda(-P_{fn})$, $\alpha_2 = \lambda(-B_p)$, $\beta_2 = \beta_1, \gamma_2 = \lambda(-Q_n)$, and λ and Λ denote the minimum and maximum eigenvalues, respectively. Let p_1 and p_2 be the solutions to $e_1(p) = 0$ and $e_2(p) = 0$, respectively. Then $p_1 \geq 0$ and $p_2 \leq 0$ because 0 is a lower solution and an upper solution to $e_1(p) = 0$ and $e_2(p) = 0$, respectively. Denote by E the unit $n \times n$ matrix. It follows that $\mathcal{E}(p_1E) \leq \mathcal{E}_{11}(p_1E) \leq e_{12}(p_1)E = 0$ and $\mathcal{E}(p_2E) \geq \mathcal{E}_{21}(p_2E) \geq e_{22}(p_2)E = 0$. So (p_1E, p_2E) is a pair of upper-lower solutions to (1), which implies that $I_m(\mathcal{E}) \supset I_m(\mathcal{E}_1) \cap I_m(\mathcal{E}_2) \supset I_m(e_1) \cap I_m(e_2)$. As an example, consider (1) with $A = \text{diag}[1, 0]$, $B = \text{diag}[1, -1]$, $Q = \text{diag}[1, 1]$ and $P_f = \text{diag}[-1, 1]$. It is easily seen that $I_m(\mathcal{E}_1) \cap I_m(\mathcal{E}_2) = (-.549306, 0)$ and $I_m(e_1) \cap I_m(e_2) = (-.5, 0)$, which provide fairly good estimates for $I_m(\mathcal{E})$, which is $(-.623225, 0)$. Note that $I_m(e_1) \cap I_m(e_2)$ can be explicitly evaluated if (3) has constant coefficients.

References

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