Estimation of Maximal Existence Intervals for Solutions to a Riccati Equation via an Upper-Lower Solution Method

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Abstract

Using an upper-lower solution method, we give estimates for the maximal existence intervals of solutions to classical Riccati differential equations that arise in linear-quadratic regular problems, differential games, risk-sensitive control and H^∞-control. Some of the results presented here are new while some are generalizations of existing results established by other methods.

1. Introduction

In this paper we obtain estimates of the maximal existence interval of the solution to the differential Riccati equation

$$P' + A^T P + PA + \Pi(P) + Q - PBP = 0, t \le t_f, P(t_f) = P_f, \tag{1}$$

which arises in many problems including linear quadratic regulator problems, differential games, H^∞-control, and risk-sensitive control. Here P' denotes the derivative of P and A^T denotes the transpose of A while A, B, Q are bounded and measurable $n \times n$ matrix functions on $(-\infty, t_f]$, and Π is a linear map that arises from the presence of state-dependent and/or jump noise in the state equation; see [1] [2] [9] [13] [14] [12]. Equation (1) plays a central role in these problems because the existence of its solution in an interval I determines the solutions to these problems in horizon I. It is well-known that the solution $P(t)$ to (1) may "blow-up" at some point t^*; that is, $P(t)$ satisfies the equation (1) on $(t^*, t_f]$ but $P(t)$ does not have a limit as $t \to t^*$, $t > t^*$. Obviously, $(t^*, t_f]$ is the maximal existence interval of the solution to (1). For a given interval I, equation (1) has a solution on I if and only if $I \subset (t^*, t_f]$. The purpose of this paper is to prove an upper-lower solution theorem and to estimate the interval $(t^*, t_f]$ by comparison method for differential equations.

The definitions and an interpretation (Theorem 1) of upper and lower solutions are given in Section 2. Section 2 also contains an upper-lower solution theorem (Theorem 4), a necessary and sufficient condition for existence of the solution to (1) on a given interval (Corollary 5), and the relationship between the maximal existence intervals of comparable equations (Proposition 6). In Section 3, we obtain several estimates for the maximal existence interval of (1) by constructing comparison equations of different types. In Section 4, we use the upper-lower solution theorem to

*This auther wishes to acknowledge the support of a Caterpillar Fellowship.

prove monotonicity of the solution to (1) and give a necessary and sufficient condition for the existence of a solution to the algebraic equation associated with (1).

Equation (1) with $\Pi = 0$ has been extensively studied; see [1] [3] [4] [8][10] [11] [16] and the references therein. Equation (1) with the term Π has been studied in several papers including [15] [11] [5] [7] [6]. This paper appears be the first use of upper and lower solutions in studies of equation (1), and it appears to be more natural and more direct way to address many issues of Riccati equations. For example, the common (often natural) assumption that $Q \geq 0$, $P_f \geq 0$ in almost all of the existing works can be replaced by that there exists a lower solution, which is much weaker. The results in this paper are either new or are generalizations of existing results established using other methods.

2. Theorems of Comparison and Upper-Lower Solutions

Notation. Denote by $\mathbb{S}^n$ the set of all real symmetric $n \times n$ matrices. We use $G \geq H$ to denote that $G - H$ is positive semidefinite and $G > H$ to denote that $G - H$ is positive definite. For $\Pi : \mathbb{S}^n \to \mathbb{S}^n$ we write $\Pi \geq 0$ if $\Pi(P) \geq 0$ for every $P \geq 0$. In this section we assume

$$\begin{cases} I = [t_0,\, t_f] \text{ or } (-\infty, t_f] \text{ is arbitrary but fixed,} \\ A \in L^\infty(I, R^{n\times n}), B, Q \in L^\infty(I, \mathbb{S}^n),\ \Pi \in L^\infty(\mathbb{S}^n, \mathbb{S}^n), \Pi \geq 0. \end{cases} \tag{2}$$

For a subspace X of a matrix space (e.g, of $R^{n\times k}$ with the inner product $\langle A, B\rangle = \mathrm{tr}(A^T B)$), denote by $L^\infty(I, X)$ the space of all bounded and measurable matrix functions from I to X. Similarly, $L^{1,\infty}(I, X)$ denotes the space of all matrix functions $P \in L^\infty(I, X)$ with $P' \in L^\infty(I, X)$. For simplicity, we will write "$Q \geq 0$" or "$Q \geq 0$ in I" for "$Q(t) \geq 0$ for every $t \in I$". For convenience we use

$$\mathrm{Ric}(P) \equiv \mathrm{Ric}(A, B, Q, \Pi; P) = A^T P + PA + \Pi(P) + Q - PBP.$$

Thus equation (1) becomes $P' + \mathrm{Ric}(P) = 0$.

Definition 1. Suppose $P \in L^{1,\infty}(I; \mathbb{S}^n)$.
P is an *upper solution* to (1) in I if

$$P' + \mathrm{Ric}(P) \leq 0,\ t \in I;\ P(t_f) \geq P_f.$$

P is a lower *solution* to (1) in I if

$$P' + \mathrm{Ric}(P) \geq 0,\ t \in I;\ P(t_f) \leq P_f.$$

P is a *solution* if it is both an upper solution and a lower solution. An upper or lower solution is *strict* if at least one of the inequalities in the definition is strict. When $A, B, Q,$ and Π are all time-invariant, an upper solution and a lower solution to the algebraic equation $\mathrm{Ric}(P) = 0$ can be defined in similar way.

Remark 1. In literature, a problem with $Q, P_f \geq 0$ is sometimes referred as a *standard problem.* Note that $Q, P_f \geq 0$ is equivalent to that $P = 0$ is a lower solution to (1). Correspondingly, $Q, P_f \leq 0$ is equivalent to that $P = 0$ is an upper solution to (1).

An upper solution and a lower solution can be interpreted meaningfully in terms of the problems where the equation arises. As an example, assume that $B = FF^T - GG^T$ and $\Pi(P) = C^T PC$ for some $F, G, C \in L^\infty(I; R^{n\times n})$ in (1) and consider the following zero-sum differential game problem:

Example Problem 1. Fix $(s,z) \in I \times R^n$. Let W be a standard Brownian motion on a probability space over $[s, t_f]$ with $W(s) = 0$ almost surely. Let $\mathcal{UW}[s,t_f]$ be the set of admissible pairs (u,w) of R^n-valued, square integrable processes adapted with the σ-field generated by $W(\cdot)$. For $(u,w) \in \mathcal{UW}[s,t_f]$, let x be the solution to the following state equation

$$dx = (Ax + Fu + Gw)dt + CxdW,\; s \le t \le t_f;\; x(s) = z. \tag{3}$$

Define the cost $J(u,w)$ as

$$J(u,w) = E\left\{x^T(t_f)P_f x(t_f) + \int_s^{t_f}(x^TQx + u^Tu - w^Tw)dt\right\},$$

where $E\{\}$ represents the expectation of the enclosed random variable. The problem is to find $(u^\infty, w^\infty) \in \mathcal{UW}[s,t_f]$ such that $\mathrm{Max}_w\mathrm{Min}_u J(u,w) = J(u^\infty, w^\infty)$. Note that if $C = 0$, u, w and x are non-random, and W and E are then irrelevant. For this problem we have

Theorem 1.
(i) *If P is a lower solution to (1) in $[s,t_f]$, then for each w, $J(u,w)$ is bounded from below.*
(ii) *If P is an upper solution to (1) in $[s,t_f]$ then for each u,$J(u,w)$ is bounded from above.*
(iii) *If P is a solution to (1) in $[s,t_f]$, then there exists (u^∞, w^∞) such that $J(u^\infty, w^\infty) = \mathrm{Max}_w\mathrm{Min}_u J(s,u,w) = z^TP(s)z$.*

Proof. (i) Suppose $P \in L^{1,\infty}([s,t_f], \mathbb{S}^n)$ and $(u,w) \in \mathcal{UW}[s,t_f]$. Let x be the solution to (3). By the Fundamental Theorem of calculus and Ito's formula applied to $x^T(t)P(t)x(t)$ we obtain

$$\begin{aligned} E\{x^T(t_f)P(t_f)x(t_f)\} - z^TP(s)z &= E\int_s^{t_f}\frac{d}{dt}x^T(t)P(t)x(t)dt \\ &= E\int_s^{t_f}[x^T(P' + A^TP + PA + C^TPC)x + 2u^TF^TPx + 2w^TG^TPx]dt. \end{aligned} \tag{4}$$

Adding (4) to $J(u,w)$ and completing the square, we obtain

$$\begin{aligned} &J(u,w) - z^TP(s)z + E\{x^T(t_f)(P(t_f) - P_f)x(t_f)\} \\ &= E\int_s^{t_f}x^T(P' + A^TP + PA + C^TPC + Q)xdt \\ &\quad + E\int_s^{t_f}(2u^TF^TPx + 2w^TG^TPx + u^Tu - w^Tw)dt \\ &= E\int_s^{t_f}x^T(P' + A^TP + PA + C^TPC + Q - PBP)xdt \\ &\quad + E\int_s^{t_f}(u + F^TPx)^T(u + F^TPx)dt - E\int_s^{t_f}(w - G^TPx)^T(w - G^TPx)dt. \end{aligned}$$

In case (i), we have that $P(t_f) \le P_f$, $P' + \mathrm{Ric}(P) \ge 0$ and the last but one integral ≥ 0.

Therefore,

$$J(u,w) \geq z^T P(s) z - E\int_s^{t_f} (w - G^T Px)^T (w - G^T Px) dt,$$

which is bounded with fixed w. Case (ii) is proved similarly. In case (iii), we have

$$\begin{aligned} &J(u,w) - z^T P(s) z \\ &= E\int_s^{t_f} \left[(u + F^T Px)^T (u + F^T Px) - (w - G^T Px)^T (w - G^T Px) \right] dt \end{aligned}$$

Denote by x^∞ the solution to (3) with $u = -F^T Px$ and $w = G^T Px$. Denote the corresponding u and w by u^∞ and w^∞, respectively. It follows that for every w, we have

$$\begin{aligned} &\mathrm{Min}_u J(u,w) \leq J(s, u^\infty, w) \\ &= z^T P(s) z - E\int_s^{t_f} (w - G^T Px)^T (w - G^T Px) dt \leq z^T P(s) z, \end{aligned}$$

which implies that $\mathrm{Max}_w \mathrm{Min}_u J(u,w) \leq z^T P(s) z$. On the other hand, it is obvious that $\mathrm{Max}_w \mathrm{Min}_u J(u,w) \geq \mathrm{Min}_u J(s,u,w^\infty) = z^T P(s) z$, and the proof of (iii) is complete.□

Proposition 2. For each $P_1, P_2 \in L^{1,\infty}(I, \mathbb{S}^n)$, we have, with $P = P_1 - P_2$,

$$\mathrm{Ric}(A,B,Q,\Pi;P_1) - \mathrm{Ric}(A,B,Q,\Pi;P_2) = \mathrm{Ric}(A - BP_2, B, 0, \Pi; P) \tag{5}$$

or equivalently,

$$\begin{aligned} &[A^T P_1 + P_1 A + \Pi(P_1) + Q - P_1 B P_1] - [A^T P_2 + P_2 A + \Pi(P_2) + Q - P_2 B P_2] \\ &= (A - BP_2)^T P + P(A - BP_2) + \Pi(P) - PBP \end{aligned}$$

Proof. This equality follows by direct verification.□

Remark 2. Equality (5) reveals an important and simple structure of equation (1). One implication is that the difference of two solutions to (1) also satisfies an equation of the same type. Another implication of (5) is that the assumption $Q, P_f \geq 0$ in most of the existing results on Riccati equations can be replaced by the existence of a lower solution, which is much weaker. In fact, if P_2 is a lower solution to (1), i.e., $Q_2 \equiv P_2' + \mathrm{Ric}(A,B,Q,\Pi;P_2) \geq 0$ and $P_2(t_f) \leq P_f$, then by (5), that P_1 is a solution to (1) if and only if $P \equiv P_1 - P_2$ satisfies

$$P' + \mathrm{Ric}(A - BP_2, B, Q_2, \Pi; P) = 0,\ P(t_f) = P_f - P_2(t_f). \tag{6}$$

Equation (6) is *standard* because $Q_2 \geq 0$ and $P_f - P_2(t_f) \geq 0$.

Proposition 3. *Suppose* $A \in L^\infty(I, R^{n\times n}), Q \in L^\infty(I, \mathbb{S}^n)$, *and* $\Pi \in L^\infty(\mathbb{S}^n, \mathbb{S}^n)$ *is linear. Then for each* $P(t_f) \in \mathbb{S}^n$ *the equation*

$$P' + A^T P + PA + \Pi(P) + Q = 0 \tag{7}$$

has a unique solution $P \in L^{1,\infty}(I, \mathbb{S}^n)$. *If* $Q \geq 0$ *in* I *and* $P(t_f) \geq 0$, *then* $P \geq 0$ *in* I. *Furthermore, if either* $Q \geq 0$ *or* $P(t_f) \geq 0$ *is strict, then* $P > 0$ *in* I.

Proof. Following the approach used in [15], let $\Phi(t,s)$ be the fundamental matrix of A, that is

$$\frac{\partial}{\partial t}\Phi(t,s) = A(t)\Phi(t,s),\ \Phi(t,t) = A(t),\ t_0 \le s,t \le t_f.$$

Note that $\Phi(t,s)^{-1} = \Phi(s,t)$ and $\frac{\partial}{\partial t}\Phi(s,t) = -A(t)\Phi(s,t)$. It follows that (7) is equivalent to

$$P(t) = \Phi(t_f,t)^T P(t_f)\Phi(t_f,t) + \int_t^{t_f}\Phi(s,t)^T\{\Pi(P(s)) + Q(s)\}\Phi(s,t)ds \qquad \text{(8b5)}$$

The Voltera equation (8) has a unique solution P, which can be found by successive approximations; say, $\{P_\nu : \nu = 0,1,2,\cdots\}$ with $P_0 = 0$. If $Q \ge 0$ in I and $P(t_f) \ge 0$, then $P_\nu \ge 0$ in I for all $\nu \ge 0$ and $t \in I$, which implies that $P = \lim_{t\to\infty} P_\nu(t) \ge 0$. If either $Q \ge 0$ in I or $P(t_f) \ge 0$ is strict, then

$$P^{\#}(t) \equiv \Phi(t_f,t)^T P(t_f)\Phi(t_f,t) + \int_t^{t_f}\Phi(s,t)^T Q(s)\Phi(s,t)ds > 0. \qquad (9)$$

It follows that $P(t) \ge P^{\#}(t) > 0$ in I. □

Theorem 4. (Upper-Lower Solution Theorem) *Suppose that* (P_1, P_2) *is a pair of upper-lower solutions to (1) in* I*. Then*
(i) $P_1 \ge P_2$ *in* I*, and* $P_1 > P_2$ *in* I *if either* P_1 *or* P_2 *is strict.*
(ii) *Equation (1) has a unique solution* P *with* $P_1 \ge P \ge P_2$ *in* I *.*

Proof. Let $P = P_1 - P_2$. Proposition 2, applied to $Q_2 \equiv P_2' + \mathrm{Ric}(P_2) - [P_1' + \mathrm{Ric}(P_1)] \ge 0$, implies that

$$P' + \mathrm{Ric}(A - BP_2, B, Q_2, \Pi; P) = 0.$$

This equation can be rewritten as a linear equation in P:

$$P' + \mathrm{Ric}\left(A - BP_2 - \frac{1}{2}BP, 0, Q_2, \Pi; P\right) = 0.$$

Note that one of P_1 and P_2 is strict implies that either $Q_2 \ge 0$ or $P(t_f) \ge 0$ is strict. Thus the results of (i) follow from Proposition 3.

For part (ii), by the local existence theory of ODE, equation (1) has a solution defined in the largest interval; say, $(\tau, t_f] \subset I$. Part (a) implies that $P_1 \ge P \ge P_2$ in $(\tau, t_f]$. Since P_1 and P_2 are bounded in I, we have that $P(\tau+) = \lim_{t\to\tau+} P(t)$ exists and $P_1(\tau) \ge P(\tau+) \ge P_2(\tau)$. If $t_f > \tau > t_0$, then the local existence theory of ODE again shows that the solution P could be further extended to the left of τ. This would contradict the definition of τ. Therefore, $\tau = t_0$ and $P(t)$ exists in $[t_0, t_f]$.□

A simple case of Theorem 4(a) is that if $Q, P_f \ge 0$ then the solution P to (1) satisfies $P \ge 0$ because 0 is a lower solution to (1).

Corollary 5. *Equation (1) has a solution if and only if it has a pair of upper-lower solutions.*

Proof. If P is a solution, then (P, P) is a pair of upper-lower solutions. The converse is part of Theorem 4 (b).$\square$

Theorem 6. *Suppose for* $i = 1, 2$, A_i, B_i, Q_i, Π_i *also satisfy (2). Let*

$$\mathcal{E}(P) \equiv P' + \textit{Ric}(A, B, Q, \Pi; P),$$
$$\mathcal{E}_i(P) \equiv P' + \textit{Ric}(A_i, B_i, Q_i, \Pi_i; P).$$

Suppose that P_1, $P_2 \in L^{1,\infty}(I, \mathbb{S}^n)$ *satisfy*

$$\mathcal{E}(P_1) \leq \mathcal{E}_1(P_1) \leq 0 \leq \mathcal{E}_2(P_2) \leq \mathcal{E}(P_2) \text{ in } I, \ P_1(t_f) \geq P_f \geq P_2(t_f) \tag{10}$$

Then $P_1 \geq P_2$ *in* I *and* $\mathcal{E}(P) = 0$ *has a solution in* I *with* $P(t_f) = P_f$.

Proof. The assumptions imply that (P_1, P_2) is a pair of upper-upper solutions of $\mathcal{E}(P) = 0$. By Theorem 4, $P_1 \geq P_2$ in I and $\mathcal{E}(P) = 0$ has a solution with $P(t_f) = P_f$. $\square$

3. Maximal Existence Intervals of a Riccati Equation

In this section we assume that A, B and Q satisfy (2) with $I = (-\infty, t_f]$. We will give estimates for maximal existence interval for the solution to (1).

Denote by $I_{\max}(\mathcal{E})$ the maximal existence interval of the equation $\mathcal{E}(P) = 0$ with given $P(t_f)$. By Theorem 4,

$$I_{\max}(\mathcal{E}) \supset \text{Dom}(P_1) \cap \text{Dom}(P_2) \tag{11ak}$$

for each pair (P_1, P_2) of upper-lower solutions to (1), where $\text{Dom}(P_1)$ and $\text{Dom}(P_2)$ are the domains of P_1 and P_2 respectively. So each pair (P_1, P_2) of upper-lower solutions to (1) gives an estimate for $I_{\max}(\mathcal{E})$ and the approximation can be as accurate as desired if (P_1, P_2) can be found to be close enough to the solution to (1).

On the other hand, Proposition 6 implies that

$$I_{\max}(\mathcal{E}) \supset I_{\max}(\mathcal{E}_1, \mathcal{E}_2), \tag{12}$$

where $I_{\max}(\mathcal{E}_1, \mathcal{E}_2) \equiv I_{\max}(\mathcal{E}_1) \cap I_{\max}(\mathcal{E}_2)$. We now use (12) to estimate $I_{\max}(\mathcal{E})$ by constructing comparison equations $(\mathcal{E}_1, \mathcal{E}_2)$ satisfying the conditions in Proposition 6. We will use the following fact.

Proposition 7. *If* $B \in \mathbb{S}^n$, *then* $B = B_p - B_n$, *where* B_p, $B_n \geq 0$ *and* $B_p B_n = 0$.

Proof. We write $B = T^T R T$, where T satisfies $T^T T = E$ and $R = \begin{pmatrix} R_1 & 0 \\ 0 & -R_2 \end{pmatrix}$ is diagonal with $R_1, R_2 \geq 0$. Define $B_p = T^T \begin{pmatrix} R_1 & 0 \\ 0 & 0 \end{pmatrix} T$ and $B_n = T^T \begin{pmatrix} 0 & 0 \\ 0 & R_2 \end{pmatrix} T$. Then $B = B_p - B_n$, B_p, $B_n \geq 0$ and $B_p B_n = 0$.$\square$

First we consider the following matrix equations:

$$\mathcal{E}_{11}(P) = P' + A^T P + PA + \Pi(P) + Q_p + PB_n P = 0, P(t_f) = P_{fp} \tag{13ca}$$

$$\mathcal{E}_{21}(P) = P' + A^T P + PA + \Pi(P) - Q_n - PB_p P = 0, P(t_f) = -P_{fn} \tag{14}$$

where (Q_p, Q_n), (B_p, B_n) and (P_{fp}, P_{fn}) are decompositions of Q, B and P_f, respectively, as described in Proposition 7. Note that for all $P \in \mathbb{S}^n,\ \mathcal{E}_{21}(P) \le \mathcal{E}(P)\ \le \mathcal{E}_{11}(P)$. So (12) implies that $I_{\max}(\mathcal{E}) \supset I_{\max}(\mathcal{E}_{11}, \mathcal{E}_{21})$.

Now let $\lambda(\,\cdot\,)$ and $\Lambda(\,\cdot\,)$ denote the minimum and maximum eigenvalues of a matrix, respectively, and construct scalar comparison equations:

$$e_{12}(p) \equiv p' + \alpha_1 p^2 + \beta_1 p + \gamma_1 = 0,\ p(t_f) = p_{1f}, \tag{15}$$
where $p_{1f} = \Lambda(P_{fp}),\ \alpha_1 = \Lambda(B_n),\ \beta_1 = \Lambda(A + A^T + \Pi(E)), \gamma_1 = \Lambda(Q_p)$

$$e_{22}(p) \equiv p' + \alpha_2 p^2 + \beta_2 p + \gamma_2 = 0, p(t_f) = p_{2f}, \tag{16}$$
where $p_{2f} = \lambda(-P_{fn}),\ \alpha_2 = \lambda(-B_p),\ \beta_2 = \Lambda(A + A^T + \Pi(E)), \gamma_2 = \lambda(-Q_n)$

Note that the solution to (13) is positive semidefinite because 0 is a lower solution to (13). Similarly, the solution to (15) is non-negative. For each scalar function $p \ge 0$, it is easy to see that $P = pE$ satisfies $\mathcal{E}(P) \le \mathcal{E}_{11}(P) \le\ e_{12}(p)E$. Similarly, the solution to (14) is negative semidefinite because 0 is an upper solution to (14) and the solution to (16) is non-positive. For each scalar function $p \le 0$, $P = pE$ satisfies that $\mathcal{E}(P) \ge \mathcal{E}_{21}(P) \ge e_{22}(p)E$. By (12), we have that $I_{\max}(\mathcal{E}) \supset I_{\max}(\mathcal{E}_{11}, \mathcal{E}_{21}) \supset I_{\max}(e_{12}, e_{22})$.

Finally we construct scalar comparison equations with constant coefficients:

$$e_{13}(p) \equiv p' + \alpha_{13} p^2 + \beta_{13} p + \gamma_{13} = 0,\ p(t_f) = p_{1f} \tag{17}$$

$$e_{23}(p) \equiv p' + \alpha_{23} p^2 + \beta_{23} p + \gamma_{23} = 0,\ p(t_f) = p_{2f} \tag{18}$$

where $\alpha_{13},\ \beta_{13}$ and γ_{13} are the upper bounds of $\alpha_1,\ \beta_1$ and γ_1, respectively, and $\alpha_{23},\ \beta_{23}$ and γ_{23} are the lower bounds of $\alpha_1,\ \beta_1$ and γ_1, respectively. Note that $e_{13}(p) \ge e_{12}(p)$ for every $p \ge 0$, and $e_{23}(p) \le e_{22}(p)$ for every $p \le 0$. Therefore, by (12), we have

$$I_{\max}(\mathcal{E}) \supset I_{\max}(\mathcal{E}_{11}, \mathcal{E}_{21}) \supset I_{\max}(e_{12}, e_{22}) \supset I_{\max}(e_{13}, e_{23}). \tag{19}$$

Theorem 8. *Relationships (19) give estimates for the maximal existence interval $I_{\max}(\mathcal{E})$ for the solution to (1).*

Example Problem 2. We illustrate the above construction by a simple example with

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, B = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, Q = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \Pi = 0, P_f = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Here $B_p = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix},\ B_n = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix},\ Q_p = Q,\ Q_n = 0,\ P_{fp} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix},\ P_{fn} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix},$ $(\alpha_1, \beta_1, \gamma_1, p_{1f}) = (\alpha_{13}, \beta_{13}, \gamma_{13}, p_{1f}) = (1, 2, 1, 1)$; $(\alpha_2, \beta_2, \gamma_2, p_{2f}) = (\alpha_{23}, \beta_{23}, \gamma_{23}, p_{2f}) = (-1, 0, 0, -1)$. It is easy to find the explicit solutions. We have

$P = \text{diag}[f(t), g(t)]$, where $f(t) = \left[e^{2\sqrt{2}t} - 1\right] / \left[1 - \sqrt{2} + \left(1 + \sqrt{2}\right)e^{2\sqrt{2}t}\right]$,

$g(t) = [1 - \sin(2t)]/\cos(2t),\ I_{\max}(\mathcal{E}) = (-.623225, 0)$;

$P_{11} = \text{diag}\left[\left(e^{-2t} - 1\right)/2, g(t)\right],\ P_{21} = \text{diag}[f(t), -t], I_{\max}(\mathcal{E}_{11}, \mathcal{E}_{21}) = (-.623225, 0)$;

$p_{12} = p_{13} = (1 - 2t)/(1 + 2t),\ p_{22} = p_{23} = -1; I_{\max}(e_{12}, e_{22}) = I_{\max}(e_{13}, e_{23}) = (-.5, 0)$

Note that the estimates are fairly close to the maximal existence interval $I_{\max}(\mathcal{E})$.

$I_{\max}(e_{13}, e_{23})$ can be calculated explicitly. We have

Proposition 9. *The solution $p_{13}(t)$ to (17) blows up at finite time t_{13}^* if and only if*
(i) $p_{1f} > r_{1,2} \equiv \left(-\beta_{13} \pm \sqrt{\beta_{13}^2 - 4\alpha_{13}\gamma_{13}}\right)/(2\alpha_{13})$ *and*

(ii)
$$I_{13} \equiv \int_{p_{1f}}^{\infty} \frac{dp}{\alpha_{13}p^2 + \beta_{13}p + \gamma_{13}} < \infty.$$

In this case, $t_f - t_{13}^* = I_{13}$. Similarly, the solution $p_{23}(t)$ to (18) blows up at finite time t_{23}^* if and only if
(iii) $p_{2f} < r_{3,4} \equiv \left(-\beta_{23} \pm \sqrt{\beta_{23}^2 - 4\alpha_{23}\gamma_{23}}\right)/(2\alpha_{23})$ *and*

(iv)
$$I_{23} \equiv \int_{p_{2f}}^{-\infty} \frac{dp}{\alpha_{23}p^2 + \beta_{23}p + \gamma_{23}} < \infty.$$

In this case, $t_f - t_{23}^* = I_{23}$.

Proof. Suppose p_{13} blows up at $t_{13}^* \in (-\infty, t_f)$. Since $p_{13}(t) \geq 0$ and monotone (see Theorem 11 below), $p_{13}(t) \to \infty$ as $t \to t_{13}^* +$. Furthermore, p_{1f} cannot be bounded from above by any constant solution of (17); otherwise $p_{13}(t)$ would be bounded. So (i) must hold. It follows that $\alpha_{13}p^2 + \beta_{13}p + \gamma_{13} > 0$ for all $p > p_{1f}$, which implies that the improper integral in (ii) must be finite. In this case, equation (17) can be written as

$$\frac{dp}{\alpha_{13}p^2 + \beta_{13}p + \gamma_{13}} = -dt,\ t \in (t_{13}^*,\ t_f).$$

Integrating from t_{13}^* to t_f, we have

$$\int_{p_{1f}}^{\infty} \frac{dp}{\alpha_{13}p^2 + \beta_{13}p + \gamma_{13}} = t_f - t_{13}^*. \tag{20}$$

Conversely if (i) and (ii) hold, then we define t_{13}^* by (20). Note that $\alpha_{13} \geq 0$ cannot be 0 in this case (otherwise the integral will not be finite). So $\alpha_{13} > 0$, which implies that $\alpha_{13}p^2 + \beta_{13}p + \gamma_{13} > 0$ for $p > p_{1f}$. Note that the solution $p_{13}(t)$ to (17) satisfies

$$\int_{p_{1f}}^{p_{13}(t)} \frac{dp}{\alpha_{13}p^2 + \beta_{13}p + \gamma_{13}} = t_f - t.$$

From this it is easy to see that $p_{13}(t) \to \infty$ as $t \to t_{13}^* +$ .This completes the proof about p_{13}. The conclusions about p_{23} can be proved similarly.□

Corollary 10. *Suppose t^* is the conjugate point of equation (1), then $t_f - t^* \geq$ min$\{I_{13}, I_{23}\}$. In other words, min$\{I_{13}, I_{23}\}$ is the minimum horizon on which equation (1) has a solution.*

4. Infinite-Horizon and Algebraic Riccati Equations

In this section we assume that A, B and Q are all constant matrices. We prove results on the monotonicity of the solution to (1) and a necessary and sufficient condition for the existence of a solution to the algebraic equation associated with (1).

Theorem 11 (Monotonicity of Solution)**.** *Suppose P is the solution of (1) in $(t^*, t_f]$. Then we have*
(a) Ric$(P_f) \geq 0$ if and only if P is increasing in $(t^*, t_f]$ as t decreases.
(b) Ric$(P_f) \leq 0$ if and only if P is decreasing in $(t^*, t_f]$ as t decreases.

Proof. If *Ric*$(P_f) \geq 0$, then P_f is a lower solution to (1). By Theorem 4, $P(t) \geq P_f$ for all $t \in (t^*, t_f]$. Fix a number $\tau > 0$ and consider the function $P_*(t) \equiv P(t - \tau)$, $t \in (t^* + \tau, t_f + \tau]$. $P_*(t)$ is also a solution to (1) with $P_*(t_f) = P(t_f - \tau) \geq P_f = P(t_f)$. By Theorem 4 again, $P_*(t) \geq P(t)$, or equivalently, $P(t - \tau) \geq P(t)$, for all $t \in (t^* + \tau, t_f]$ and every $\tau > 0$. So $P(t)$ is increasing in $(t^*, t_f]$ as t decreases. This proves (a). Part (b) is proved similarly by using the fact that P_f is an upper solution.□

We now prove a necessary and sufficient condition for the existence of solutions to

$$A^T P + PA + \Pi(P) + Q - PBP = 0 \tag{21}$$

in the form of an intermediate value theorem; that is, between upper and lower constant solutions to (21), there exists at least one solution.

Theorem 12. *Equation (21) has a solution $P \in \mathbb{S}^n$ if and only if there are $P_{1f}, P_{2f} \in \mathbb{S}^n$, $P_{1f} \geq P_{2f}$ such that $\mathcal{E}(P_{1f}) \leq 0$, $\mathcal{E}(P_{2f}) \geq 0$.*

Proof. The necessity is obvious. For sufficiency, consider equation (1) with $P_f = P_{if}$, $i = 1, 2$. Because P_{1f} is an upper solution and P_{2f} is a lower solution to (1) in $(-\infty, t_f]$, by Theorem 4, there exist solutions $P_i(t)$ in $(-\infty, t_f]$ such that $P_{1f} \geq P_1(t) \geq P_2(t) \geq P_{2f}$. By Theorem 11, both P_1 and P_2 are monotone. It follows that for $i = 1, 2$, $P_i^* = \lim_{t \to -\infty} P_i(t)$ exist, and both P_1^* and P_2^* are constant solutions to (21). □

In fact, P_1^* and P_2 are respectively the maximal and minimal solutions to (21) between P_{1f} and P_{2f}. The properties of solutions to (21) and their relationship to stability and detectability in terms of upper and lower solutions will be discussed in a separate paper.

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