Linear Cumulant Control and Its Relationship to Risk-Sensitive Control

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Abstract

Matrix differential equation descriptions of the cumulants of an integral quadratic cost associated with a linear system with white-noise input were derived in the mid-70s using generalized Karhunen-Loeve expansion techniques. Here, these same descriptions are derived directly from the cumulant generating function of the cost. A generalization of the k-cumulant control problem class introduced in 1998 is also presented. The solution to this more general class of optimal cumulant control problems is given, and the risk sensitive control problem of optimizing the cumulant generating function of the LQG cost is shown to be included in this cumulant control class.

1 Introduction

In 1998 Pham, Liberty and Sain introduced a general class of Linear-Quadratic-Gaussian (LQG) control problems in which the objective is minimization of a performance index that is a finite, linear combination of cumulants of integral quadratic cost over linear, memoryless, full-state-feedback control laws [9]. The formulation of this "k-cumulant" optimization problem utilizes a coupled matrix differential equation description of the cumulants of an integral quadratic form (IQF) in a Gaussian process. These equations were first derived by Liberty and Hartwig [8] in 1976. In that work the Gaussian process is the state of a linear system with white-noise input. The work reported in [8] evolved from that of Liberty [7] in 1971 where it was observed that all cumulants of such IQFs are quadratic-affine in the mean of the system initial state. Two examples of control problems in the k-cumulant class for $k = 1$ and $k = 2$ respectively are the classical minimum mean LQG problem and the Minimum Cost Variance (MCV) problem [12, 13].

Since the early 1970s other researchers have developed the theory of risk-sensitive control; see for example [2, 3, 4, 5, 6, 14]. Also see [16] where Won, Sain and Spencer presented a brief history of risk-sensitive control and pointed out the close relationship between risk-sensitive control and "the notion of optimal cost cumulants, in particular cost variance." In this paper we demonstrate an even deeper relationship between risk-sensitive and cumulant control.

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In the next section we demonstrate that the coupled matrix differential equations presented in [8] and utilized in [9] can be generated directly from the well known quadratic-affine form of the cumulant generating function of the (IQF) cost. In section 3 we consider a further generalization of k-cumulant control, which we simply refer to as cumulant control. Observed formally by Won [15], the generalization consists of replacing the finite linear combination of cost cumulants in the performance index with an infinite series of cost cumulants. The solution to this "cumulant control" problem is introduced in section 3. Finally, in section 4 we demonstrate the relationship between the solution to the risk-sensitive control problem (for the linear, memoryless, full-state-feedback case) and the solution to the cumulant control problem.

2 Cumulant Descriptions

Consider the linear system

$$dx = Fxds + Gdw,\ x(t_0) = x_0 \tag{1}$$

where the matrix functions $F(s) \in R^{n\times n},\ G(s) \in R^{n\times p}$ are continuous, $w(s) \in R^p$ is the standard Wiener process, and x_0 is non-random. For every pair (t_0, x_0), associate with (1) a cost

$$J(t_0, x_0) = x^T(t_f)\,Q_f x(t_f) + \int_{t_0}^{t_f} x^T(s)\,N(s)\,x(s)\,ds \tag{2}$$

where $Q_f \in R^{n\times n}$ is a constant matrix and ≥ 0, $N(t) \in R^{n\times n}$ is a continuous matrix and ≥ 0. Here J is considered to be a function of t_0 and x_0. In the following we will replace $(t_0,\ x_0)$ by $(t, x) \in [t_0, t_f] \times R^n$, and $J(t_0, x_0)$ by $J(t, x)$, the "cost-to-go", which is (2) with t_0 replaced by t and $x(s)$ replaced by the solution of (1) with initial condition $x(t) = x$.

It is well-known (see [1] for example) that the moment-generating and cumulant-generating functions of the cost-to-go

$$\phi(\theta, t, x) = E\{e^{\theta J(t,x)}\}\ \text{ and }\ \psi(\theta, t, x) = \ln\phi(\theta, t, x)\,,$$

where $E\{\cdot\}$ is the expected value of the enclosed random variable, can be expressed in terms of the solution of a particular Riccati equation as follows:

Theorem 1. *For fixed θ, let the functions $\rho(\theta, t)$ and $S(\theta, t)$ be solutions to the equations*

$$\begin{cases} S' + F^T S + SF + 2SWS + \theta N = 0 \\ S(\theta,\ t_f) = \theta Q_f, \end{cases} \tag{3}$$

and

$$\begin{cases} \rho' = -\rho\, tr(SW)\,, \\ \rho(\theta,\ t_f) = 1, \end{cases} \tag{4}$$

where $W = GG^T$ and $(')$ denotes the derivative. Then

$$\begin{array}{ll} (a) & \phi(\theta, t, x) = \rho(\theta, t)\, exp\left(x^T S(\theta, t)\, x\right) \\ (b) & \psi(\theta, t, x) = d(\theta, t) + x^T S(\theta, t)\, x \end{array}$$

with $d(\theta,t) = ln\ \rho(\theta,t)$ *satisfying*

$$\begin{cases} d' = -\,tr\,(SW)\,, \\ d\,(\theta,\ t_f) = 0. \end{cases}$$

□

Recall that for $i = 1, 2, \cdots$ the moments $m_i(t,x)$ of $J(t,x)$ are defined as the coefficients of $\theta^i/i!$ in the McLaurin series expansion of the moment-generating function $E\{e^{\theta J}\}$, while the cumulants $\kappa_i(t,x)$ of $J(t,x)$ are the coefficients of $\theta^i/i!$ in the McLaurin series expansion of the cumulant-generating function

$$\ln E\{e^{\theta J}\} = \sum_{i=1}^{\infty} \frac{\theta^i}{i!}\kappa_i. \tag{5}$$

We now present an alternative and more direct derivation of the cumulant expressions contained in [8].

Theorem 2. *The cumulants* $\kappa_i(t,x)$ *of* $J(t,x)$ *can be expressed as*

$$\kappa_i(t,x) = d_i(t) + x^T H_i(t)\,x$$

where d_i *and* H_i *satisfy the following Lyapunov-type differential equations:*

$$H_1' + F^T H_1 + H_1 F + N = 0,\ H_1(t_f) = Q_f,$$

$$H_i' + F^T H_i + H_i F + 2\sum_{j=1}^{i-1} \frac{i!}{j!\,(i-j)!} H_j W H_{i-j} = 0,\ H_i(t_f) = 0,\ i \geq 2,$$

$$d_i' = -tr\,(H_i(t)\,W)\,,\ d_i(t_f) = 0,\ i \geq 1.$$

□

Proof. Utilizing the functions in Theorem 1 define

$$H_i(t) = \frac{\partial^i}{\partial\theta^i} S(0,t) \text{ and } d_i(t) = \frac{\partial^i}{\partial\theta^i}\rho(0,t)\,.$$

Then by expanding the cumulant generating function expression of Theorem 1.b in a McLaurin series we get

$$\psi(\theta,t,x) = \rho(\theta,t) + x^T(t)\,S(\theta,t)\,x = \sum_{i=1}^{\infty} \left(d_i(t) + x^T H_i(t)\,x\right)\frac{\theta^i}{i!}.$$

Matching coefficients with the series in (5) we observe that the i^{th} cumulant of $J(t,x)$ can be expressed as

$$\kappa_i(t,x) = d_i(t) + x^T H_i(t)\,x.$$

The equation for H_1 follows by evaluating the $\frac{\partial}{\partial\theta}$ of (3) at $\theta = 0$ and using the fact $S(0,t) = 0$. For $i \geq 2$, by evaluating the $\frac{\partial^i}{\partial\theta^i}$ of (3) at $\theta = 0$ and using $S(0,t) = 0$, we derive the differential equations and boundary conditions satisfied by $H_i(t) = S^{(i)}(0,t)$. Similarly from (4) we get the equations $d_i' = -\mathrm{tr}(H_i(t)\,W)$ and $d_i(t_f) = 0$. □

3 Cumulant Control Problems

We now consider a general cumulant control problem associated with the following linear control system and integral quadratic cost function:

$$dx_c = (Ax_c + Bu)\,ds + Gdw,\ s \in [t_0, t_f]\,;\ x_c(t_0) = x_0, \tag{6}$$

$$J_c = x_c^T(t_f)\,Q_f x_c(t_f) + \int_{t_0}^{t_f} \left(x_c^T Q x_c + u^T R u\right) ds \tag{7}$$

where the matrix functions $A(t) \in R^{n\times n}$, $B(t) \in R^{n\times m}$, $G(t) \in R^{n\times p}$ are continuous. $Q_f \in R^{n\times n}$ is constant matrix and ≥ 0, $Q(t) \in R^{n\times n}$ and $R(t) \in R^{m\times m}$ are continuous on $[t_0, t_f]$, $Q(t) \geq 0$ and $R(t) > 0$. We will assume that u is a linear memoryless, full-state-feedback control given by

$$u(s) = K(s)\,x(s)\,,$$

with feedback gain $K(s) \in R^{m\times n}$. Then the system (6) and the cost (7) can be written as

$$dx_c = Fx_c ds + Gdw,\ x_c(t_0) = x_0, \tag{8}$$

$$J_c(t_0, x_0, K) = x_c^T(t_f)\,Q_f x_c(t_f) + \int_{t_0}^{t_f} x_c^T(s)\,N(s)\,x_c(s)\,ds, \tag{9}$$

where

$$F(s) = A(s) + B(s)\,K(s)\,,\ N(s) = Q(s) + K^T(s)\,R(s)\,K(s)\,. \tag{10}$$

It is well-known that cumulants and moments are expressible in terms of each other, however, cumulants are more useful quantities in stochastic optimal control because of their common quadratic-affine form.

For a sequence of real numbers $\mu = \{\mu_1, \mu_2, \cdots\}$, we can use (10) and the cumulant expressions in Theorem 2 to define a series of cumulants

$$\kappa(t_0, x_0) = x_0^T \sum_{i=1}^{\infty} \mu_i H_i(t_0) x_0 + \sum_{i=1}^{\infty} \mu_i d_i(t_0) \tag{11}$$

Without loss of generality, we will assume that $\mu_1 = 1$.

Define $\mathcal{H}(s) = (H_1(s)\,, H_2(s)\,, \cdots)$ and $\mathcal{D}(s) = (d_1(s)\,, d_2(s)\,, \cdots)$ and rewrite the equations in Theorem 2, for H_i and d_i as

$$\begin{cases} H_1' + \mathcal{F}_1(\mathcal{H}, K) = 0,\ H_1(t_f) = Q_f, \\ H_i' + \mathcal{F}_i(\mathcal{H}, K) = 0,\ H_i(t_f) = 0,\ i \geq 2, \\ d_i' + \mathcal{G}_i(\mathcal{H}) = 0,\ d_i(t_f) = 0,\ i \geq 1. \end{cases} \tag{12}$$

where

$$\begin{cases} \mathcal{F}_1(\mathcal{H}, K) = F^T H_1 + H_1 F + N, \\ \mathcal{F}_i(\mathcal{H}, K) = F^T H_i + H_i F + 2\sum_{j=1}^{i-1} \frac{i!}{j!(i-j)!} H_j W H_{i-j}, i \geq 2, \\ \mathcal{G}_i(\mathcal{H}) = \operatorname{tr}(H_i(s)\,W)\,. \end{cases} \tag{13}$$

Now let

$$\Phi(t_0, \mathcal{H}(t_0), \mathcal{D}(t_0); x_0; \mu) = \kappa(t_0, x_0).$$

The cumulant control problem that we wish to solve is

$$\begin{array}{c} Min \\ K \in \mathcal{K}_\mu \end{array} \Phi(t_0, \mathcal{H}(t_0), \mathcal{D}(t_0); x_0; \mu)$$

subject to (12) and (13). The class, $\mathcal{K}_\mu$ of admissible feedback gains is the set of all $m \times n$ matrix functions that are continuous on $[t_0, t_f]$ with values in a compact subset of the vector space of all $m \times n$ real matrices, and which yield existence of solutions on $[t_0, t_f]$ to (12) and in combination with the given μ_i's yield a convergent series in (11).

Remark: *For a given stochastic linear system are there μ-sequences that result in a rich class K_μ of admissible function gains? The answer to this question is affirmative. Due to space limitation here we present some results without proof. It can be shown that there is a $\sigma > 0$ such that if $\{\mu_i^*\}_{i=1}^{\infty}$ where $\mu_i^* = i!\mu_i$ satisfies*

$$0 \le \mu_i^* \le \sigma \mu_j^* \mu_{i-j}^* \tag{14}$$

for $i = 2, \dots$ and $j = 1, \dots, i-1$, then K_μ contains many admissible feedback gains (for example, it contains all K's with small norm) with convergence of the series (11).

To illustrate further that there are many μ_i^*'s satisfying (14), consider the following. For a given $\sigma > 0$, there are intervals $\{[a_i, b_i]\}_{i=1}^{\infty}$ such that if $\{\mu_i^*\}_{i=1}^{\infty}$ is a sequence with $\mu_i^* \in [a_i, b_i]$, then it satisfies the condition (14). For example, let $a_i = \alpha\theta^{i-1}$ and $b_i = \sigma\alpha^2\theta^{i-2}$ for some numbers α, $\theta > 0$ such that $\sigma\alpha > \theta$ and $\mu_i^* \in [a_i, b_i]$. Then for $i = 2, 3, \cdots$ and $j = 1, \cdots, i-1$,

$$\frac{\mu_i^*}{\mu_j^* \mu_{i-j}^*} \le \frac{b_i}{a_j a_{i-j}} = \sigma.$$

A particular example satisfying (14) with $\sigma = 1$ is $\mu_i = \theta^i / i!$, which is the risk-sensitive case; see (5).

We prove

Theorem 3. *The admissible feedback gain K that minimizes $\Phi(t_0, \mathcal{H}(t_0), \mathcal{D}(t_0); x_0; \mu)$ and the corresponding $H_1, H_2, \dots$ and $d_1, d_2, \dots$ satisfy the following equations*

$$\begin{cases} F = A + BK, \\ K = -R^{-1}B^T \sum_{i=1}^{\infty} \mu_i H_i, \\ H_1' + F^T H_1 + H_1 F + Q + K^T R K = 0, \; H_1(t_f) = Q_f, \\ H_i' + F^T H_i + H_i F + 2\sum_{j=1}^{i-1} \frac{i!}{j!(i-j)!} H_j W H_{i-j} = 0, \; H_i(t_f) = 0, \; i \ge 2, \\ d_i' = -tr(H_i(s) W), \; d_i(t_f) = 0, \; i \ge 1. \end{cases}$$

□

Proof. The control problem is a standard Mayer type problem.

For $t \in [t_0, t_f]$, consider (13) on the interval $[t_0, t]$ with terminal values $H_i(t) = y_i$ and $d_i(t) = z_i$. From the representation of κ, we look for a value function of the following form

$$V(t, y) = x_0^T \sum_{i=1}^{\infty} \mu_i (y_i + h_i(t)) x_0 + \sum_{i=1}^{\infty} \mu_i (z_i + e_i(t))$$

where $h_i(t)$'s and $e_i(t)$'s are functions to be determined. By the Verification Theorem [4, Thms 4.1, 4.4], $V(t,y)$ has to satisfy

$$\underset{K}{Min}\left\{-\frac{\partial V}{\partial t}+\sum_{i=1}^{\infty}\frac{\partial V}{\partial y_i}\mathcal{F}_i+\sum_{i=1}^{\infty}\frac{\partial V}{\partial z_i}\mathcal{G}_i\right\}=0. \tag{15}$$

Substituting (13) into (15) we obtain

$$\underset{K}{Min}\left\{x_0^T\left(-\sum_{i=1}^{\infty}\mu_i h_i'(t)+\sum_{i=1}^{\infty}\mu_i\mathcal{F}_i(y,K)\right)x_0+\sum_{i=1}^{\infty}\mu_i\left(-e_i'(t)+\mathcal{G}_i(y)\right)\right\}=0 \tag{16}$$

Note that among all of the terms in $\{\cdots\}$, only $\sum_{i=1}^{\infty}\mu_i\mathcal{F}_i(y,K)$ depends on K. We have

$$\begin{aligned}\sum_{i=1}^{\infty}\mu_i\mathcal{F}_i(y,K)= \; & (A+BK)^T\left(\sum_{i=1}^{\infty}\mu_i h_i\right)+\left(\sum_{i=1}^{\infty}\mu_i h_i\right)(A+BK)+Q+\\ & K^TRK+2\sum_{i=2}^{\infty}\mu_i\sum_{j=1}^{i-1}\frac{i!}{j!\,(i-j)!}H_jWH_{i-j}\end{aligned}$$

This matrix quadratic is minimal when $K=-R^{-1}B^T\sum_{i=1}^{\infty}\mu_i h_i$. With this choice of K, let $H_i(t)$ and $d_i(t)$ be the solutions to (13) with $H_i(t)=y_i$ and $d_i(t)=z_i$. Take

$$h_i(t)=H_i(t_0)-H_i(t)\,,\; e_i(t)=d_i(t_0)-d_i(t)\,,$$

then the minimum in (16) will be zero. It turns out that the value function $V(t,y)$ is

$$V(t,y)= \; x_0^T\sum_{i=1}^{\infty}\mu_i H_i(t_0)\,x_0+\sum_{i=1}^{\infty}\mu_i d_i(t_0)\,,$$

which is the optimal value of $\Phi(t_0,\mathcal{H}(t_0),\mathcal{D}(t_0);x_0;\mu)$. □

4 Relationship Between Linear Cumulant Control and Risk-Sensitive Control

The existence of optimal risk-sensitive control is well-known [5, 14]. Denote the optimal value

$$\Psi(\theta,\,t,x)= \underset{K\in\mathcal{K}_\mu}{Min}\left[\ln E\{e^{\theta J_c(t,x,K)}\}\right],$$

where $\theta>0$ is assumed.

Theorem 4. *The value function $\Psi(\theta,\,t,x)$ can be expressed as*

$$\Psi(\theta,\,t,x)=x^TS(\theta,\,t)\,x+d(\theta,\,t)\,,$$

where $S(\theta,\, t)$ and $d(t,\theta)$ satisfy the Riccati system

$$\begin{cases} K = -(\theta R)^{-1} B^T S, \\ S' + (A+BK)^T S + S(A+BK) + 2SWS + \theta\left(Q + K^T RK\right) = 0, \\ S(\theta, t_f) = \theta Q_f, \\ d' = -tr(SW),\; d(\theta, t_f) = 0. \end{cases} \tag{17}$$

□

We answer a long-standing question about the relationship between risk-sensitive and cumulant control.

Theorem 5. *Suppose A, B, Q, R, W are given matrices as in Theorem 4. Suppose the K, $H_1 \ldots, H_n, \ldots,$ and the constant θ satisfy the following system*

$$\begin{cases} H_1' + (A+BK)^T H_1 + H_1(A+BK) + Q + K^T RK = 0,\; H_1(t_f) = Q_f, \\ H_i' + (A+BK)^T H_i + H_i(A+BK) + 2\sum_{j=1}^{i-1} \frac{i!}{j!(i-j)!} H_j W H_{i-j} = 0, \\ H_i(t_f) = 0,\; i \geq 2, \\ K = -(\theta R)^{-1} B^T \sum_{i=1}^{\infty} \frac{\theta^i}{i!} H_i. \end{cases} \tag{18}$$

Then the matrix

$$\mathcal{H} = \sum_{i=1}^{\infty} \frac{\theta^i}{i!} H_i$$

satisfies the matrix equation (17)

$$\mathcal{H}' + (A+BK)^T \mathcal{H} + \mathcal{H}(A+BK) + \theta\left(Q + K^T RK\right) + 2\mathcal{H}W\mathcal{H} = 0 \tag{19}$$

and K is a solution to

$$\Psi(\theta,\, t, x) = \begin{matrix} Min \\ K \end{matrix} \;\left[ln\;\; E_{tx} e^{\theta J(t,x,K)}\right],$$

where

$$J(t,x,K) = \int_t^{t_f} \left(x^T Q x + u^T R u\right) ds + x^T(t_f)\, Q_f x(t_f).$$

In other words, $\mathcal{H}$ is the solution to the risk-sensitive control problem. □

Proof. Multiply the i-th equation in (18) by $\frac{\theta^i}{i!}$ for $i \geq 1$ and add them up. We get

$$\mathcal{H}' + (A+BK)^T \sum_{i=1}^{\infty} \frac{\theta^i}{i!} H_i + \sum_{i=1}^{\infty} \frac{\theta^i}{i!} H_i (A+BK)$$
$$+\, \theta\left(Q + K^T RK\right) + 2\sum_{i=2}^{\infty}\sum_{j=1}^{i-1} \frac{\theta^j}{j!} H_j W \frac{\theta^{i-j}}{(i-j)!} H_{i-j} = 0.$$

Since

$$\sum_{i=2}^{\infty}\sum_{j=1}^{i-1} \frac{\theta^j}{j!} H_j W \frac{\theta^{i-j}}{(i-j)!} H_{i-j} = \sum_{j=1}^{\infty} \frac{\theta^j}{j!} H_j W \sum_{k=1}^{\infty} \frac{\theta^k}{k!} H_k = \mathcal{H}W\mathcal{H},$$

we obtain

$$\mathcal{H}' + (A+BK)^T \mathcal{H} + \mathcal{H}(A+BK) + \theta\left(Q + K^T RK\right) + 2\mathcal{H}W\mathcal{H} = 0.$$

Note that this is precisely (19) with $K = -(\theta R)^{-1} B^T \mathcal{H}$. Therefore $\mathcal{H}$ is a solution to the risk-sensitive control problem. □

References

[1] A. Bensoussan, *Stochastic Control of Partially Observable Systems,* London, Cambridge University Press, 1992.

[2] W. H. Fleming and D. Hernandez-Hernandez, "Risk-Sensitive Control of Finite State Machines on an Infinite Horizon II," *SIAM Journal of Control Optimization*, Vol. 37, No. 4, pp. 1048-1069, 1999.

[3] W. H. Fleming and W. M. McEneaney, "Risk-Sensitive Optimal Control and Differential Games", *Lecture Notes in Control and Information Sciences*, Vol. 184, pp. 185–197, Springer-Verlag, Berlin, 1992.

[4] W. H. Fleming and R. W. Rischel, *Deterministic and Stochastic Optimal Control,* Springer-Verlag, 1975.

[5] D. H. Jacobson, "Optimal Stochastic Linear Systems with Exponential Criteria and Their Relation to Deterministic Differential Games," *IEEE Transactions on Automatic Control*, Vol. 18, pp. 124-131, 1973.

[6] M. R. James, "Asymptotic Analysis of Nonlinear Stochastic Risk-Sensitive Control and Differential Games," *Mathematics Control Signals Systems 5,* pp. 401-417, 1992.

[7] S. R. Liberty, "Characteristics Functions of LQG Control," *Ph.D. Dissertation*, Dept. of Electrical Engineering, University of Notre Dame, 1971.

[8] S. R. Liberty and R. C. Hartwig, "On the Essential Quadratic Nature of LQG Control Performance Measure Cumulants", *Information and Control,* Vol. 32, No. 3, pp. 276-305, 1976.

[9] K. D. Pham, S. R. Liberty, and M. K. Sain, "Linear Optimal Cost Cumulant Control: A k-Cumulant Problem Class," *Proceedings Thirty-Sixth Allerton Conference on Communication, Control, and Computing*, pp. 460-469, October 1998.

[10] K. D. Pham, S. R. Liberty, and M. K. Sain, "Evaluating Cumulant Controllers on a Benchmark Structure Protection Problem in the Presence of Classic Earthquakes," *Proceedings Thirty-Seventh Allerton Conference on Communication, Control, and Computing*, pp. 617-626, October 1999.

[11] M. K. Sain and S. R. Liberty, "Performance Measure Densities for a Class of LQG Control Systems," *IEEE Transactions on Automatic Control*, AC-16, No. 5, pp. 431-439, October 1971.

[12] M. K. Sain, "Control of Linear Systems According to the Minimal Variance Criterion: A New Approach to the Disturbance Problem," *IEEE Transactions on Automatic Control*, AC-11, No. 1, pp. 118-122, January 1966.

[13] M. K. Sain, C. H. Won, B. F. Spencer, Jr., and S. R. Liberty, "Cumulants and Risk-Sensitive Control: A Cost Mean and Variance Theory with Application to Seismic Protection of Structures," pp. 427-459 in *Advances in Dynamic Games and Applications*, Annals of the International Society of Dynamic Games, Vol. 5, Jerzy A Filar, Vladimir Gaitsgory, and Koichi Mizukami, Editors. Boston: Birkhauser, 2000.

[14] P. Whittle, *Risk-Sensitive Optimal Control*, New York: John Wiley & Sons, 1990.

[15] C. H. Won, "Cost Cumulants in Risk-Sensitive and Minimal Cost Variance Control," *Ph.D. Dissertation*, Dept. of Electrical Engineering, University of Notre Dame, 1994.

[16] C. H. Won, M. K. Sain, and B. F. Spencer, Jr.,"Relationships in Modern Stochastic Control: Risk-Sensitive and Minimal Cost Variance Control," *Proceedings Allerton Conference on Communication, Control, and Computing,* pp. 343-352, 1996.