Multiple Solutions and Regularity of H-systems

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April 1994

Abstract

The main result of this paper proves the existence of multiple solutions to a class of generalized constant mean curvature equations, called H-systems. Also contained is a regularity for conformal n-harmonic maps.

1 Introduction

In this paper, we consider some systems of the form

$$\mathrm{div}(|\nabla u|^{n-2}\nabla u) = f(u, \nabla u), \tag{1}$$

where $u \in W^{1,n}(\Omega, R^k)$, $n,\ k \geq 2$; $\Omega \subset R^n$ is a bounded smooth domain, and $f : R^k \times R^{nk} \to R^k$ is a smooth function. We assume

$$|f(u, \nabla u)| \leq \Lambda |\nabla u|^n, \tag{2}$$

for some constant $\Lambda > 0$ that may depend on u.

A well-known example of (1) is the n-harmonic map equation. Let $(N, h) \hookrightarrow R^k$ be a C^∞ compact Riemannian submanifold. An *n-harmonic map* $u : \Omega \to N$ is a critical point of the n-energy $\int_\Omega |\nabla u|^n dx$ in the space of functions $u \in W^{1,n}(\Omega, R^k)$ with $u(x) \in N$ for a.e. $x \in \Omega$. The equation for n-harmonic maps is

$$\mathrm{div}(|\nabla u|^{n-2}\nabla u) = |\nabla u|^{n-2} Q(u, \nabla u), \tag{3}$$

where $Q(u, \cdot)$ is the trace of the second fundamental form of N at $u(x) \in N$; $Q(u, \nabla u)$ is quadratic in ∇u.

There is a vast literature on the regularity and partial regularity of solutions to harmonic (or p-harmonic) map type equations; see [4][11][13][15][17][19][20][24][26][30] and other references therein.

Our interest in this paper is mainly on the H-systems in higher dimensions. Suppose $u \in W^{1,n}(\Omega, R^{n+1}),\ u = (u^1, ..., .u^{n+1})$. Then the cone generated by the image $u(\Omega)$, with vertex being the origin of R^{n+1}, has a well-defined *volume*

$$V(u) = \frac{1}{n+1} \int_\Omega u \cdot u_1 \wedge \cdots \wedge u_n;$$

see [24]. Here $u_1 \wedge \cdots \wedge u_n$ is the cross product of the partial derivatives $u_1, ..., u_n$, which can be described as follows. For any vector $v \in R^{n+1}$,

$$v \cdot u_1 \wedge \cdots \wedge u_n = \begin{vmatrix} v^1 & v^2 & ... & v^{n+1} \\ u_1^1 & u_1^2 & ... & u_1^{n+1} \\ ... & ... & ... & ... \\ u_n^1 & u_n^2 & ... & u_n^{n+1} \end{vmatrix}.$$

Consider the minimization problem

$$\min \int_\Omega |\nabla u|^n,\ u = \eta \text{ on } \partial\Omega,\ V\ (u) = c, \tag{4}$$

for a given $\eta \in W^{1,n}(\Omega, R^{n+1})$ and a constant c. A critical point of (4) is called an n-harmonic map with prescribed volume; it satisfies

$$\text{div } \left(|\nabla u|^{n-2} \nabla u\right) = H u_1 \wedge \cdots \wedge u_n,\ u = \eta \text{ on } \partial\Omega, \tag{5}$$

where H is the Lagrange multiplier.

When $n = 2$, (5) becomes

$$\Delta u = H u_1 \wedge u_2. \tag{6}$$

A conformal solution of (6) represents a surface of constant mean curvature; see, e.g., [28] [31]. The existence of solutions and multiple solutions of (6) were established in many works, including [6] [21] [27] [29] [31] [33]. In Theorems 5 and 12 below, we prove for relatively small H and boundary data, there is a solution of least energy-the *small solution*, and there is a *large solution,* with the same boundary data. This generalizes the early work of Hildebrandt [21], Brezis and Coron [6] and Struwe [29] for $n = 2$.

For the regularity of (2-)harmonic maps u on a domain $\Omega \subset R^2$ (or a smooth surface), Heléin [19] proved their C^∞ regularity. Assuming u is conformal, or stationary or energy minimizing, Morrey[23], Grüter[16] and Schoen[25] established the regularity of u earlier. For the H-system (6) with constant H, Wente [31] showed that any solution of (6) is analytic. Grüter [16] proved the $C^{1,\alpha}$ regularity ($0 < \alpha < 1$) of conformal solutions to (6), where H may depend on u; same result was obtained later by Bethuel [5] assuming that $|DH(u)|$ is bounded. Wente's result was generalized to (5) in [10][24], which implies that all solutions of (5) are $C^{1,\alpha}$ regular. In this paper, we prove the $C^{1,\alpha}$ regularity of conformal solutions to (1), which generalizes the work of Grüter [16]. In particular, conformal n-harmonic maps from $\Omega \subset R^n$ (or an n-manifold) are $C^{1,\alpha}$, and conformal solutions of (5) with bounded $H = H(u)$ are also $C^{1,\alpha}$. Unlike in two dimension, one cannot reparametrize a solution to obtain conformality; so the conformality condition for solutions to (1) is fairly strong. It is conjectured that all n-harmonic maps and solutions to (5) with bounded $H = H(u)$ are $C^{1,\alpha}$. Generally speaking, $C^{1,\alpha}$ regularity is optimal for solutions of (1)as shown by examples in [22].

2 Existence of Solutions to H-systems

For any $u \in W^{1,n}(\Omega, R^{n+1})$, the image $u(\Omega)$ is a generalized "hypersurface" with area

$$A(u) = \int_\Omega J(u)dx, \ \ J(u) = |u_1 \wedge \cdots \wedge u_n|,$$

where $J(u)$ is the Jacobian of u. Note that

$$\begin{aligned} |v \cdot u_1 \wedge \cdots \wedge u_n| &\leq |v||u_1| \cdots |u_n| \\ &\leq |v| \left(\frac{|u_1|^2 + \cdots + |u_n|^2}{n} \right)^{n/2} \\ &= |v| \frac{|\nabla u|^n}{\sqrt{n^n}}, \end{aligned} \tag{7}$$

and the equalities hold if and only if u is conformal. Here we say that a function $u \in W^{1,n}(\Omega, R^k)$ is *conformal* if for some function $\lambda(x)$ and all $i, j = 1, ..., n$,

$$u_i \cdot u_j = \lambda(x)\delta_{ij} \, . \tag{8}$$

It follows from (7)

$$|u_1 \wedge \cdots \wedge u_n| \leq \frac{|\nabla u|^n}{\sqrt{n^n}} \text{ and } A(u) \leq \frac{1}{\sqrt{n^n}} \int_\Omega |\nabla u|^n \, , \tag{9}$$

and each of the equalities holds iff u is conformal.

We now discuss some properties of the volume functional V.

First note that if $u = (u^1, ..., u^{n+1})$ and $u^1 = 0$ on $\partial\Omega$, then for all $i = 1, ..., n$,

$$\int_\Omega u^1 \frac{\partial\,(u^2, ..., u^{n+1})}{\partial\,(x^1, ..., x^n)} = (-1)^{i-1} \int_\Omega u^i \frac{\partial(u^1, ..., \overset{\wedge}{u^i}, ..., u^{n+1})}{\partial\,(x^1, ..., x^n)}, \tag{10}$$

and the volume V can be written as

$$V(u) = \int_\Omega u^1 \frac{\partial\,(u^2, ..., u^{n+1})}{\partial\,(x^1, ..., x^n)}. \tag{11}$$

In fact, (10) follows by expanding the determinant $\frac{\partial\left(u^2, ..., u^{n+1}\right)}{\partial(x^1, ..., x^n)}$ in the i-th column, using integration by parts together with the fact

$$\sum_{\alpha=1}^{n} \frac{\partial}{\partial x^\alpha} \frac{\partial(u^2, ... \, , \overset{\wedge}{u^i}, ..., u^{n+1})}{\partial(x^1, ..., \hat{x^\alpha}, ..., x^n)} = 0.$$

Expanding $V(u)$ in terms of $u^1, ..., u^{n+1}$ we get (11) by using (10).

As a consequence of (10) and isoperimetric inequality, we have

Proposition 1 *If $u = (u^1, ..., u^{n+1}) \in W^{1,n}(\Omega, R^{1+n})$ and $u^1 = 0$ on $\partial\Omega$, then for some constant C_1,*

$$\left| \int u^1 \frac{\partial\,(u^2, ..., u^{n+1})}{\partial\,(x^1, ..., x^n)} \right| \leq \; C_1 \left\| \nabla u^1 \right\|_{L^n(\Omega)} \cdots \left\| \nabla u^{n+1} \right\|_{L(\Omega)} \tag{12}$$

Proof: We may assume that none of u^i is constant (otherwise, the inequality is trivial), and that $\|\nabla u^i\|_{L^n(\Omega)} = 1$ for all i (by the homogeneity of (12) in u^i). Then (9) implies

$$A(u) \leq \frac{1}{\sqrt{n^n}} \int_\Omega |\nabla u|^n = \left(\frac{n+1}{n}\right)^{n/2}.$$

Denote $v = (0, u^2, ..., u^{n+1})$. Then $A(v) \leq A(u)$ and $V(v) = 0$. So

$$\left| \int_\Omega u^1 \frac{\partial (u^2, ..., u^{n+1})}{\partial (x^1, ..., x^n)} \right| = |V(u) - V(v)|$$

is the volume enclosed by the graphs of u and v, whose area is $A(u) + A(v)$. By isoperimetric inequality (see [2], for example),

$$|V(u) - V(v)| \leq \frac{1}{C} [A(u) + A(v)]^{(n+1)/n},$$

where $C = (n+1)\omega_n^{\frac{1}{n}}$ and ω_n is the area of the unit n-sphere S^n. Therefore, for an absolute constant C_1,

$$\left| \int_\Omega u^1 \frac{\partial (u^2, ..., u^{n+1})}{\partial (x^1, ..., x^n)} \right| \leq C_1,$$

which shows (12) . □

(12) implies the following corollaries.

Corollary 2 *If $u \in W^{1,n}(\Omega, R^{n+1})$ and $u^i|\partial\Omega = 0$ for some $i = 1, ..., n+1$, then the functional V is continuous at u in the norm of $W^{1,n}(\Omega, R^{n+1})$.*

Corollary 3 *Suppose $u, v \in W^{1,n}(\Omega, R^{n+1})$ and $v = 0$ or $u = 0$ on $\partial\Omega$, then for some constant C,*

$$\left| \int_\Omega v \cdot u_1 \wedge \cdots \wedge u_n \right| \leq C \|\nabla v\|_{L^n(\Omega)} \|\nabla u\|^n_{L(\Omega)}. \tag{13}$$

Proof: Expand $|\int_\Omega v \cdot u_1 \wedge \cdots \wedge u_n|$ in terms of $v^1, ..., v^{n+1}$ and apply (12) to each term. □

We now derive a useful property of

$$R(v,u)=\int_\Omega v\cdot u_1\wedge ...\wedge u_n.$$

Suppose $u,v,w\in W^{1,n}(\Omega,R^{n+1})$, $w=0$ or $v=0$ on $\partial\Omega$, and $u_t=u+tw$ for $0\le t\le 1$. For a moment, suppose that $u,v,w\in C^2$. Then

$$\begin{aligned}
R(v,u+w)-R(v,u) &= \int_\Omega v\cdot (u_t)_1\wedge ...\wedge (u_t)_n \,|_0^1 \\
&= \int_\Omega \sum_{i=0}^n v\cdot (u_t)_1\wedge ...\wedge \underset{i}{w_i}\wedge ...\wedge (u_t)_n \\
&= -\int_\Omega\int_0^1 \sum_{i=0}^n w_i\cdot (u_t)_1\wedge ...\wedge \underset{i}{v}\wedge ...\wedge (u_t)_n \\
&= \int_\Omega\int_0^1 \sum_{i=0}^n w\cdot (u_t)_1\wedge ...\wedge \underset{i}{v_i}\wedge ...\wedge (u_t)_n \\
&\quad \sum{}_{j\neq i}\sum\nolimits_{i=0}^n w\cdot (u_t)_1\wedge ...\wedge \underset{j}{(u_t)_{ji}}\, ...\wedge \underset{i}{v}\wedge ...\wedge (u_t)_n \\
&= \int_\Omega\int_0^1 \sum_{i=0}^n w\cdot (u_t)_1\wedge ...\wedge \underset{i}{v_i}\wedge ...\wedge (u_t)_n\,.
\end{aligned} \tag{14}$$

Here we used the skew-symmetry of the cross product, which implies the term $\sum_{j\neq i}\sum_{i=0}^n\cdots=0$. It follows

$$|R(v,u+w)-R(v,u)|\le C\,||w||_\infty\,||\nabla v||_{L^n}\,|||\nabla u|+|\nabla w|||_{L^n}^{n-1}\,; \tag{15}$$

or

$$|R(v,u+w)-R(v,u)|\le C\,||\nabla v||_\infty\,||w||_{L^n}\,|||\nabla u|+|\nabla w|||_{L^n}^{n-1}\,. \tag{16}$$

The estimates (15) and (16) show that, in addition to the condition that $u,v,w\in W^{1,p}(\Omega,R^{n+1})$, it is enough to assume $w\in C^0$ for (15) to hold, and $v\in W^{1,\infty}(\Omega,R^{n+1})$ for (16) .

Applying (14) to $u=0$ and $v,w\in W^{1,n}(\Omega,R^{n+1})$ with v or $w=0$ on $\partial\Omega$, we obtain

$$\begin{aligned}
\int_\Omega v\cdot w_1\wedge\cdots\wedge w_n &= \int_\Omega\int_0^1 t^{n-1}dt\sum_{i=0}^n w\cdot w_1\wedge ...\wedge \underset{i}{v_i}\wedge ...\wedge w_n \\
&= -\frac{1}{n}\sum\nolimits_{i=1}^n\int_\Omega v_i\cdot w_1\wedge\cdots\wedge \underset{i}{w}\wedge\cdots\wedge w_n.
\end{aligned} \tag{17}$$

The equation (5) can be derived by using (17) . We only need to calculate $\frac{d}{dt}V\ (u+t\phi)$ for any $\phi \in W_0^{1,n}(\Omega, R^{n+1})$. By (17)

$$\begin{aligned} & \frac{d}{dt}V\ (u+t\phi) \\ = & \frac{1}{n+1}\int_\Omega \phi \cdot u_1 \wedge \cdots \wedge u_n + \frac{1}{n+1}\sum_{i=1}^n \int_\Omega u \cdot u_1 \wedge \cdots \wedge \underset{i}{\phi_i} \wedge \cdots \wedge u_n \\ = & \frac{1}{n+1}\int_\Omega \phi \cdot u_1 \wedge \cdots \wedge u_n - \frac{1}{n+1}\sum_{i=1}^n \int_\Omega \phi_i \cdot u_1 \wedge \cdots \wedge \underset{i}{u} \wedge \cdots \wedge u_n \\ = & \int_\Omega \phi \cdot u_1 \wedge \cdots \wedge u_n. \end{aligned}$$

□

The following is another property of R that we prove by (17) .

Theorem 4 *Suppose that, as* $m \to \infty$, $u^m \rightharpoonup u$ *in* $W_0^{1,n}(\Omega, R^{n+1})$, *and either* $v^m \to v$ *in* $W^{1,n}(\Omega, R^{n+1})$ *or* $||v^m - v||_\infty \to 0$ *with* v *being continuous, then*

$$R(v^m, u^m) \equiv \int v^m \cdot u_1^m \wedge \ldots \wedge u_n^m \to R(v,u), \text{ as } m \to \infty.$$

Proof: By (13) and the assumptions, we have

$$|R(v^m,u^m) - R(v,u^m)| \le \left\{ \begin{array}{c} \|\nabla v^m - \nabla v\|_{L^n(\Omega)} \\ \text{or} \quad \|v^m - v\|_\infty \end{array} \right\} \|\nabla u^m\|_{L^n(\Omega)}^n \to 0$$

as $m \to \infty$. This implies that we may assume $v^m \equiv v$. Furthermore, we may assume that v is C^2 by approximating v by smooth functions in the norm of $W^{1,n}$, and in the norm of C^0 in case v is continuous.

Now, because $u^m \rightharpoonup u$ in $W_0^{1,n}(\Omega, R^{n+1})$, we have $u^m \to u$ in L^n. By (17) ,

$$\begin{aligned} R(v,u^m) &= -\frac{1}{n}\sum\nolimits_{i=1}^n \int_\Omega v_i \cdot (u^m)_1 \wedge \cdots \wedge \underset{i}{u^m} \wedge \cdots \wedge (u^m)_n \\ &\to -\frac{1}{n}\sum\nolimits_{i=1}^n \int_\Omega v_i \cdot u_1 \wedge \cdots \wedge \underset{i}{u} \wedge \cdots \wedge u_n, \text{ as } m \to \infty \\ &= \int_\Omega v \cdot u_1 \wedge \cdots \wedge u_i \wedge \cdots \wedge u_n = R(v,u). \end{aligned}$$

□

We now prove the existence of the small solutions.

Theorem 5 *Suppose $\eta \in W^{1,n}(\Omega, R^{n+1})$ and $0 \neq H$ is a constant satisfying*

$$||\eta||_\infty |H| \leq \sqrt{n^n}. \tag{18}$$

Then the problem (5) *has a solution u that satisfies $||u||_\infty \leq ||\eta||_\infty$.*

Remark 6 *The case $n = 2$ of this theorem is due to Hildebrandt [21]; see also [20][27][31][32][10]. In the next section, we will show that if η and H are small enough, then the problem* (5) *has another "big" solution.*

Remark 7 *In general, a bound condition for H like* (18) *is needed for the existence of a solution. Consider the case when Ω is the unit ball and $\eta(x) = (x, 0)$ for $x \in \partial\Omega$. If H satisfies* (18)*, then a conformal representation of a sphere cap of radius $r = \sqrt{n^n}/|H| \geq 1$ with $u|\partial\Omega = \eta$ is a solution to* (4)*. If $|H| > \sqrt{n^n}$, it can be shown that* (4) *has no solution.*

Proof of Theorem5: Note that the equation in (5) is the Euler-Lagrange equation of the functional I, defined by

$$I(u) = \int_\Omega |\nabla u|^n + \frac{nH}{n+1} u \cdot u_1 \wedge ... \wedge u_n, \tag{19}$$

without constraint. Since I is neither bounded from above, nor from below, it has no global maximum nor minimum. We will find a local minimum of I by minimizing I on the subset

$$M = \left\{ u \in W^{1,n}\left(\Omega, R^{n+1}\right) : u = \eta \text{ on } \partial\Omega,\ ||u||_\infty |H| \leq \sqrt{n^n} \frac{2n+1}{2n} \right\}.$$

It is easy to see that M is weakly closed and convex subset of $W^{1,n}(\Omega, R^{n+1})$. For any $u \in M$, it follows from (9) that

$$\begin{aligned} I(u) &\geq \int_\Omega |\nabla u|^n - \frac{n|H|\,||u||_\infty}{(n+1)\sqrt{n^n}} \int_\Omega |\nabla u|^n \\ &\geq \frac{1}{2n+2} \int_\Omega |\nabla u|^n, \end{aligned} \tag{20}$$

So I is coercive. From [23] or [8], I is quasiconvex. By the Theorem II.4 in [1], I is weakly lower semicontinuous. It follows from the direct method that I has a minimum u in M.

We now show that $||u||_\infty \leq \ ||\eta||_\infty$. Suppose k is any number satisfying

$$||\eta||_\infty |H| < k |H| \leq \sqrt{n^n}\frac{2n+1}{2n}. \tag{21}$$

Let $\phi = \max\{|u|-k,0\}$. Then $\phi \in W_0^{1,n}(\Omega, R^+) \cap L^\infty$, and $u - t\phi u \in M$ for sufficiently small $t \geq 0$. It follows from the minimality of u,

$$\begin{aligned}
0 &\geq -\frac{d}{dt}|_{t=0} I(u - t\phi u) = \int_\Omega < \phi u, DI(u) > \\
&= n\int_\Omega |\nabla u|^{n-2}\nabla u \nabla(\phi u) + \frac{|H|}{n+1}(\phi u)\cdot u_1 \wedge ... \wedge u_n \\
&\geq n\int_\Omega \left(|\nabla u|^n - \frac{|H| ||u||_\infty}{n+1}|u_1 \wedge ... \wedge u_n|\right)\phi + n\int_\Omega |\nabla u|^{n-2}\nabla u\cdot u\nabla\phi \\
&\geq \frac{n}{2n+2}\int_{\{|u|>k\}} |\nabla u|^n\phi + n\int_{\{|u|>k\}} |\nabla u|^{n-2}(\nabla u\cdot u)^2 |u|^{-1}.
\end{aligned}$$

It follows that $\nabla u = 0$ a.e. on $\{|u| > k\}$, which implies that $\nabla\phi = 0$ a.e. Ω. So $\phi \equiv 0$, or $|u| \leq k$. As k in (21) is arbitrary, $||u||_\infty \leq ||\eta||_\infty$, which implies that $||u||_\infty |H| \leq ||u||_\infty |H| < \sqrt{n^n}\frac{2n+1}{2n}$. So u is an interior minimum point of M in the norm $||\cdot||_\infty$; it is then has to be a critical point of I and satisfies (5) . □

3 The Existence of Large Solutions

In Section 2, we showed that if $||\eta||_\infty |H| \leq \sqrt{n^n}$, then the Dirichlet problem (22) has a solution. In this section, we will prove that there is at least another *big* solution if η is small enough. When $n = 2$, the existence of multiple solutions of (22) was established in [6][29] under the optimal assumption $0 \neq ||\eta||_\infty |H| < 2$. The optimal condition for our case is expected to be $0 \neq ||\eta||_\infty |H| < \sqrt{n^n}$, though our proof of Theorem (12) does not yield such an estimate.

Denote by u_0 the solution we found in Theorem 5 of Section 3. We will solve the problem

$$\operatorname{div}\left(|\nabla u|^{n-2}\nabla u\right) = H u_1 \wedge \cdots \wedge u_n, \; u = \eta \text{ on } \partial\Omega, \tag{22}$$

for $u = u_0 + v$ with some $v \in W_0^{1,n}(\Omega, R^{n+1})$, $v \neq 0$. Note that (22) is the Euler-Lagrange equation of the functional

$$E(u) = \int_\Omega |\nabla u|^n + \frac{nH}{n+1} Q(u), \tag{23}$$

without constraint, where $Q(u) = \int_\Omega u \cdot\, u_1 \wedge \cdots \wedge u_n = (n+1)\, V(u)$. The method is to find a critical point of (23) . . We need some preparations.

Proposition 8 *For* $a,\ b \in R^k$, $(k \geq 2$ *an integer*$)$, *there holds*

$$|a+b|^n \quad = \quad |a|^n + |b|^n + n\,|a|^{n-2}\, a \cdot b + M(a,b) \tag{24}$$

where $M(a,b)$ satisfies

$$|M(a,b)| \leq n(n-2)\,(|a| + |b|)^{n-3}\,|a|\,|b|^2\,. \tag{25}$$

Proof: By the fundamental theorem of calculus,

$$\begin{aligned} M(a,b) &\equiv |a+b|^n - \Big(|a|^n + |b|^n + n\,|a|^{n-2}\, a\cdot b\Big) \\ &= \int_0^1 \frac{d}{dt}\,|a+tb|^n\, dt - \Big(|b|^n + n\,|a|^{n-2}\, a\cdot b\Big) \\ &= n\int_0^1 |a+tb|^{n-2}\left(a\cdot b + t\,|b|^2\right)\, dt - \Big(|b|^n + n\,|a|^{n-2}\, a\cdot b\Big) \\ &= n\int_0^1 \int_0^t \frac{d}{ds}\,|a+sb|^{n-2}\, a\cdot b\, dsdt + n\int_0^1 \int_0^1 t\frac{d}{ds}\,|sa+tb|^{n-2}\,|b|^2\, dsdt. \end{aligned} \tag{26}$$

(25) follows from the following estimate: For any $p \geq 1$,

$$\sup_{0\leq t\leq 1}\left|\frac{d}{dt}\,|a+tb|^p\right| \leq p\,(|a| + |b|)^{p-1}\,|b|\,. \tag{27}$$

□

Proposition 9

$$Q(u_0 + v) = Q(u_0) + Q(v) + \sum_{i=1}^{n-1} Q_i(v) \tag{28}$$

where $Q_i(v)$ *is homogeneous in* v *of degree* i *and homogeneous in* u_0 *of degree* $n+1-i$.

Proof: Let $g(t) = Q(u_0 + tv)$. Then (28) is the Taylor expansion of g at $t = 1$, where $Q_i(v) = g^{(i)}(0)/i!$ and $Q(v) = Q_{n+1}(v)$. □

Proposition 10

$$n \int_\Omega |\nabla u_0|^{n-2} \nabla u_0 \nabla v + \frac{nH}{n+1} Q_1(v) = 0. \tag{29}$$

Proof: The is just the weak form of the equation (22) ; v serves as a test function. □

It follows from (23)-(29) that

$$\begin{aligned} E(u_0 + v) &= \int_\Omega |\nabla u_0|^n + \frac{nH}{n+1} Q(u_0) + \int_\Omega |\nabla v|^n \\ &\quad + \frac{nH}{n+1} Q_n(v) + E_2(v) + \frac{nH}{n+1} Q(v), \end{aligned} \tag{30}$$

where

$$Q_n(v) = (n+1) \int_\Omega u_0 \cdot v_1 \wedge \cdots \wedge v_n, \text{ by } (17),$$

$$E_2(v) = \int_\Omega M(\nabla u_0, \nabla v) + \sum_{i=2}^{n-1} Q_i(v). \tag{31}$$

Since the first two terms of (30) are constant, we are led to the functional

$$\Phi(v) \equiv \int_\Omega |\nabla v|^n + \frac{nH}{n+1} Q_n(v) + E_2(v) + \frac{nH}{n+1} Q(v). \tag{32}$$

We look at each term in (32). Note that by (9),

$$|Q_n(v)| \leq C \sup |u_0| \int_\Omega |\nabla v|^n, \text{ where } C = \frac{n+1}{\sqrt{n^n}}. \tag{33}$$

The isoperimetric inequality for mappings [2][Theorem 12] implies that if $v \in W_0^{1.n}(\Omega, R^{n+1})$ then

$$|V(v)| \leq \frac{1}{C} A(v)^{\frac{n+1}{n}}, \tag{34}$$

where $C = (n+1)\omega_n^{\frac{1}{n}}$ and ω_n is the area of the unit n-sphere S^n. In terms of $Q(v) = (n+1) V(v)$ and $\int_\Omega |\nabla v|^n$, it follows from (17) that

$$|Q(v)|^{\frac{n}{n+1}} \leq \frac{1}{S} \int_\Omega |\nabla v|^n, \text{ where } S = n^{\frac{n}{2}} \omega_n^{\frac{1}{n+1}}. \tag{35}$$

To estimate $E_2(v)$, we first notice that $Q(u) = R(u,u)$ and

$$\begin{array}{ll}\sum_{i=2}^{n-1} Q_i(v) & = Q(u_0+v) - Q(u_0) - Q_1(v) - Q_n(v) - Q(v) \\ & = [R(u_0, u_0+v) - R(u_0,u_0) - R(u_0,v)] + \\ & [R(v, u_0+v) - R(v,v) - nR(u_0,v)] - (n+1)\, R(v,u_0)\end{array} \tag{36}$$

By (14) , we have

$$\begin{array}{l}|R(v, v+u_0) - R(v,v) - nR(u_0,v)| \\ = \left|\int_\Omega \int_0^1 \left[\int_0^t \frac{d}{ds} \sum_{i=0}^n u_0 \cdot (v+su_0)_1 \wedge ... \wedge \underset{i}{v_i} \wedge ... \wedge (v+su_0)_n \, ds\right] dt\right| \\ \leq C \left|\left|u_0\right|\right|_\infty \left|\left|\nabla u_0\right|\right|_{L^n} \left|\left|\nabla v\right|\right|_{L^n} \left|\left||\nabla v| + |\nabla u_0|\right|\right|_{L^n}^{n-2}.\end{array} \tag{37}$$

$$\begin{array}{l}|R(u_0, u_0+v) - R(u_0,u_0) - R(u_0,v)| \\ = \left|\int_\Omega \int_0^1 \sum_{i=0}^n u_0 \cdot (su_0+tv)_1 \wedge ... \wedge \underset{i}{v_i} \wedge ... \wedge (su_0+tv)_n \,|_{s=0}^{s=1} dt\right| \\ = \left|\int_\Omega \int_0^1 \int_0^1 \frac{d}{ds} \sum_{i=0}^n u_0 \cdot (su_0+tv)_1 \wedge ... \wedge \underset{i}{v_i} \wedge ... \wedge (su_0+tv)_n \, dsdt\right| \\ \leq C \left|\left|u_0\right|\right|_\infty \left|\left|\nabla u_0\right|\right|_{L^n} \left|\left|\nabla v\right|\right|_{L^n} \left|\left||\nabla v| + |\nabla u_0|\right|\right|_{L^n}^{n-2}.\end{array} \tag{38}$$

$$|R(v,u_0)| \leq C \left|\left|\nabla v\right|\right|_{L^n} \left|\left|\nabla u_0\right|\right|_{L^n}^n . \tag{39}$$

By (31) , (25) and (36)-(39) , we get

$$\begin{array}{ll}|E_2(v)| & \leq \int_\Omega n(n-2)\left(|\nabla u_0| + |\nabla v|\right)^{n-3} |\nabla u_0| \, |\nabla v|^2 + \left|\sum_{i=2}^{n-1} Q_i(v)\right| \\ & \leq C \int_\Omega \left(|\nabla u_0|^{n-2} |\nabla v|^2 + |\nabla u_0|^2 |\nabla v|^{n-2}\right) + \\ & \quad C \int_\Omega \sum_{i=2}^{n-1} |u_0|_\infty |\nabla u_0| \, |\nabla v| \left(|\nabla v|^{n-2} + |\nabla u_0|^{n-2}\right) + C\,|\nabla v|\,|\nabla u_0|^n \\ & \leq C_0 \int_\Omega \sum_{i=1}^{n-1} |\nabla u_0|^{n-i} |\nabla v|^i + C\,|\nabla v|\,|\nabla u_0|^n .\end{array} \tag{40}$$

Note that Φ is unbounded from above and below, and it is a typical case not satisfying the Palais-Smale conditions. The standard variational method fails to give the existence of a critical point. In the case $n = 2$, where E_2 does not appear in Φ, Brezis and Coron [6] was able to find a nontrivial critical point of Φ as a proper dilation of a minimum of $\int_\Omega |\nabla v|^2 + \frac{2H}{3} Q_2(v)$ subject to $Q(v)$ =constant. For $n \geq 3$, the terms of Φ have at least three different homogeneities, therefore, the method in [6] is unlikely to work. Our method is to apply a mountain pass theorem of Ambrosetti-Rabinowitz [3] in a min-max scheme. We will use the following form of the theorem in [3], as used by Brezis and Nirenberg[7] in solving elliptic equations with critical exponents.

Theorem 11 *[3][7]Assumption: Let Φ be a C^1 function on a Banach space E. Suppose there exists a neighborhood U of 0 in E and a constant ρ such that $\Phi(u) \geq \rho$ for every $u \in \partial U$, and*

$$\Phi(0) < \rho \text{ and } \Phi(v) < \rho \text{ for some } v \notin U.$$

Set $c = \inf_{p\in P} \max_{w\in p} \Phi(w) \geq \rho$, where P denotes the class of paths joining 0 to v.

Conclusion: There is a sequence $\{u_i\}$ in E such that $\Phi(u_i) \to c$ and

$$\Phi'(u_i) \to 0 \text{ in } E^*.$$

The advantage of this theorem is that it does not require (PS)-condition. We will show that a subsequence of $\{u_i\}$ converges to a nontrivial critical point of Φ. Our result is stated as follows.

Theorem 12 *$\eta \in W^{1,n}(\Omega, R^{n+1})$ and $||\eta||_\infty + ||\nabla\eta||_{L^n(\partial\Omega)}$ is small enough, then the problem (22) has at least two solutions.*

Remark 13 *One solution is the small solution u_0 found in Section 2; it satisfies $||u_0||_\infty \leq ||\eta||_\infty$ and is a minimum of E in M. Thus*

$$\frac{1}{2n+2}\int_\Omega |\nabla u_0|^n \leq E(u_0) \leq E(\bar{\eta}),$$

where $\bar{\eta}\,(x) = |x|\,\eta\left(\frac{x}{|x|}\right)$ is a special extension of η. Thus

$$\int_\Omega |\nabla u_0|^n \leq \int_\Omega |\nabla\bar{\eta}|^n + 2nHQ(\bar{\eta}) \leq C_0 \int_\Omega |\nabla\bar{\eta}|^n \leq C_1 \left(||\eta||_\infty + ||\nabla\eta||_{L^n(\partial\Omega)}\right).$$

It follows that $||\eta||_\infty + ||\nabla\eta||_{L^n(\partial\Omega)}$ is small implies that $||u_0||_\infty + ||\nabla u_0||_{L^n(\Omega)}$ is also small. The smallness condition used in the proof actually is referred to u_0.

We now start the proof of Theorem 12 with verifying the conditions in Theorem 11.

Proposition 14 *There are numbers $\delta, \rho > 0$ such that*

$$\Phi(v) \geq \rho \textit{ for } v \in W_0^{1,n}(\Omega, R^{n+1}) \textit{ with } ||\nabla v||_{L^n(\Omega)} = \delta.$$

Proof: By (32) , (33) , (35), (40) and the Hölder inequality, for any $\epsilon > 0$, there are numbers $C_0, C(\epsilon)$, such that

$$\begin{aligned}\Phi(v) \geq & \int_\Omega |\nabla v|^n - C_0||u_0||_\infty \int_\Omega |\nabla v|^n - \epsilon \int_\Omega |\nabla v|^n - \\ & C(\epsilon)\left(|u_0|_\infty + ||\nabla u_0||_{L^n(\Omega)}\right) - C_0 \left(\int_\Omega |\nabla v|^n\right)^{\frac{n+1}{n}}.\end{aligned}$$

Fix $\epsilon = \frac{1}{4}$ and a number $\delta > 0$ such that $C_0\delta \leq \frac{1}{8}$. Suppose u_0 satisfies $C_0||u_0||_\infty \leq \frac{1}{4}$ and $C(\frac{1}{4})\left(|u_0|_\infty + ||\nabla u_0||_{L^n(\Omega)}\right) \leq \frac{1}{16}\delta^n$. Then for any $v \in W_0^{1,n}(\Omega, R^{n+1})$ with $||\nabla v||_{L^n(\Omega)} = \delta$, we have

$$\begin{aligned}\Phi(v) \geq & \frac{1}{4}\int_\Omega |\nabla v|^n - C_0\left(\int_\Omega |\nabla v|^n\right)^{\frac{n+1}{n}} - C(\epsilon)\left(|u_0|_\infty + ||\nabla u_0||_{L^n(\Omega)}\right) \\ \geq & \frac{1}{8}\delta^n - C(\epsilon)\left(|u_0|_\infty + ||\nabla u_0||_{L^n(\Omega)}\right) \geq \frac{1}{16}\delta^n.\end{aligned}$$

The proposition holds with $\rho = \frac{1}{16}\delta^n$. □

Proposition 15 *There is a $v \in W_0^{1,n}(\Omega, R^{n+1})$ such that*

$$\begin{aligned}&\Phi(v) \leq 0, \\ &\sup_{0\leq t} \Phi(tv) < \frac{S^{n+1}}{|H|^n (n+1)}. \qquad (41)\end{aligned}$$

The proof of Proposition 15 will be given later. Now we prove Theorem 12.

Proof of Theorem 12: By the theorem of Ambrosetti-Rabinowitz above and Propositions 14 and 15, there exists $\{v^i\} \subset W_0^{1,n}(\Omega, R^{n+1})$ such that as $i \to \infty$,

$$\begin{array}{c}\Phi(v^i) = \int_\Omega \left|\nabla v^i\right|^n + \frac{nH}{n+1} Q_n\left(v^i\right) + \\ E_2(v^i) + \frac{nH}{n+1} Q(v^i) \to c,\end{array} \tag{42}$$

where

$$c = \inf_P \max_{v \in P} \{\Phi(v)\}, \tag{43}$$

and

$$\begin{aligned}\frac{1}{n}\Phi'(v^i) \quad &= - \operatorname{div}(|\nabla v^i|^{n-2} \nabla v^i) + \frac{H}{n+1} Q_n'(v^i) + \\ &\frac{1}{n} E_2'(v^i) + H v_1^i \wedge \cdots \wedge v_n^i \to 0 \text{ in } W^{-1,n'},\end{aligned} \tag{44}$$

where $n' = \dfrac{n}{n-1}$. Multiply (44) by v^i and integrate. We get

$$\int_\Omega \left|\nabla v^i\right|^n + \frac{H}{n+1} < Q_n'\left(v^i\right), v^i > + \frac{1}{n} < E_2'(v^i), v^i > + HQ(v^i) \to 0. \tag{45}$$

We claim that

$$\int_\Omega \left|\nabla v^i\right|^n \le C \tag{46}$$

for some constant C. To prove (46), we first note that since $Q_n(v^i)$ is homogeneous in v^i of degree n,

$$\left|< Q_n'\left(v^i\right), v^i >\right| = \left|nQ_n\left(v^i\right)\right| \le \frac{n(n+1)}{\sqrt{n^n}} \|u_0\|_\infty \int_\Omega \left|\nabla v^i\right|^n; \tag{47}$$

and by (40) and the Hölder inequality, for $\epsilon > 0$, there is a constant $C(\epsilon)$, such that

$$\left|E_2(v^i)\right| \le C(\epsilon) \int_\Omega |\nabla u_0|^n + \epsilon \int_\Omega \left|\nabla v^i\right|^n, \tag{48}$$

$$\left|< E_2'(v^i), v^i >\right| \le C(\epsilon) \int_\Omega |\nabla u_0|^n + \epsilon \int_\Omega \left|\nabla v^i\right|^n. \tag{49}$$

Now look at the difference of (42) and (45), we then get

$$\frac{H}{n+1} Q_n(v^i) + \frac{H}{n+1} Q(v^i) + \frac{1}{n} < E_2'(v^i), v^i > - E_2(v^i) \to -c.$$

It follows for some constant C, depending on ϵ,

$$\left|Q(v^i)\right| \leq \epsilon \int_\Omega \left|\nabla v^i\right|^n + C(\epsilon). \tag{50}$$

Combining (50) with (42) , we get (46) . As in [18], we may assume, by passing to a subsequence, that v^i weakly converges to a v in $W^{1,n}(\Omega, R^{n+1})$, and strongly converges to v in $W^{1,p}(\Omega, R^{n+1})$ for any $p \in [1, n)$.

We claim that v is nontrivial. For otherwise, $v \equiv 0$ implies that

$$\begin{aligned} < Q_n'\left(v^i\right), v^i > \; &= nQ_n\left(v^i\right) = (n+1) \textstyle\int_\Omega u_0 \cdot v_1^i \wedge \cdots \wedge v_n^i, \to 0; \\ &E_2(v^i) \to 0, < E_2'(v^i), v^i > \to 0. \end{aligned} \tag{51}$$

By passing to a subsequence if necessary, we may assume further that $\int_\Omega \left|\nabla v^i\right|^n \to l$. It follows that $Q(v^i) \to -\dfrac{l}{H}$ by (45) . By (42) , we have

$$l + \frac{nH}{n+1}\left(-\frac{1}{H}\right) l \to c. \tag{52}$$

It follows that $c = \dfrac{l}{n+1}$. On the other hand, by isoperimetric inequality,

$$l \geq S \left|\frac{l}{H}\right|^{\frac{n}{n+1}},$$

which implies $l \geq \dfrac{S^{n+1}}{|H|^n}$. Therefore,

$$c \geq \frac{S^{n+1}}{|H|^n (n+1)}.$$

This is a contradiction, because Proposition 15 implies that $c < \dfrac{S^{n+1}}{H^n(n+1)}$. So v is nontrivial. Taking the limit in (44) , we have that v satisfies $\Phi'(v) = 0$, or equivalently, $u = u_0 + v$ is a solution. □

The rest of this section is devoted to the proof of Proposition 15. The case $n = 2$ has been shown in [6].We generalize the argument in [6] to higher dimensions.

For $v \in W_0^{1,n}(\Omega, R^{n+1})$, denote

$$E_3(v) = \int_\Omega |\nabla v|^n + \frac{nH}{n+1} Q_n(v); \tag{53}$$

$$R(v) = \frac{E_3(v)}{|Q(v)|^{\frac{n}{n+1}}}; \tag{54}$$

$$S = \inf \left\{ \frac{\int_\Omega |\nabla v|^n}{|Q(v)|^{\frac{n}{n+1}}},\ Q(v) \neq 0, v \in W_0^{1,n}(\Omega, R^{n+1}) \right\}. \tag{55}$$

Define

$$J = \inf \left\{ T(v) :\ Q(v) \neq 0, v \in W_0^{1,n}(\Omega, R^{n+1}) \right\}. \tag{56}$$

We first prove

Proposition 16 *$J < S$.*

Proof: Suppose $0 \in \Omega$ and $\nabla u(0) \neq 0$. Choose a coordinate basis $e_{1,...,}e_{n+1}$ for R^{n+1} that has the same orientation as the canonical basis of R^{n+1} such that

$$\gamma \equiv \frac{\partial u}{\partial x_1}(0) \cdot e_1 + ... + \frac{\partial u}{\partial x_n}(0) \cdot e_n < 0. \tag{57}$$

Let $v : R^n \to S^n$ be the stereographic projection:

$$v(x) = \frac{(2x, -2)}{1 + |x|^2},\ x \in R^n, \tag{58}$$

(v is written in the coordinate $e_{1,...,}e_{n+1}$). For $\epsilon > 0$, consider the map

$$v^\epsilon(x) = \frac{(2x, -2\epsilon)}{\epsilon^2 + |x|^2}.$$

Let $R > 0$ be a number such that $B_{4R} \equiv B_{4R}(0) \subseteq \Omega$. Let $\xi \in C_0^1(B_{2R}, [0,1])$ be a cut-off function such that $\xi = 1$ on B_R. Note that $\xi v^\epsilon \in C_0^1(\Omega, R^{n+1})$ and the following properties of v^ϵ can be easily verified:

$$v^\epsilon(x) = \frac{1}{\epsilon} v(\frac{x}{\epsilon}),$$

$$|v^\epsilon(x)| = \frac{2}{\sqrt{\epsilon^2 + |x|^2}}, \tag{59}$$
$$|\nabla v^\epsilon(x)| \leq \frac{C}{\epsilon^2 + |x|^2},$$

for a constant C independent of ϵ and x.

We shall establish

$$T(\xi v^\epsilon) = S + c_0\epsilon + O\left(\epsilon^{1+\alpha}\right) \text{ as } \epsilon \to 0, \tag{60}$$

where $c_0 < 0$ and $\alpha \in (0,1)$ are constants. Here, as a notation, $O(f)$ denotes a quantity satisfying $|O(f)| \leq C|f|$ for some constant C. The inequality of the Proposition 16 follows by taking ϵ small enough.

We now proceed to show (60). By the mean value theorem,

$$|f(a+b) - f(a)| = O\left(\sup_{0\leq t\leq 1} |f'(a+tb)|\right)|b|. \tag{61}$$

Applying this to $f(a) = |a|^n$ with $a = \xi\nabla v^\epsilon, b = \nabla\xi v^\epsilon$, we have

$$\begin{aligned} \int_\Omega |\nabla(\xi v^\epsilon)| &= \int_{R^n} |\xi\nabla v^\epsilon + \nabla\xi v^\epsilon|^n \\ &= \int_{R^n} |\xi\nabla v^\epsilon|^n + O\left(\int_{R^n} \left(|\xi\nabla v^\epsilon| + |\nabla\xi v^\epsilon|\right)^{n-1} |\nabla\xi v^\epsilon|\right). \end{aligned} \tag{62}$$

Since v^ϵ is conformal and $v^\epsilon(R^n)$ is a sphere of radius $\frac{1}{\epsilon}$, we have

$$\int_{R^n} |\nabla v^\epsilon|^n = \sqrt{n^n} \cdot \text{area}\left(v^\epsilon(R^n)\right) = \frac{\sqrt{n^n}\omega_n}{\epsilon^n}. \tag{63}$$

On the other hand, by (59),

$$\int_{R^n} (\xi^n - 1)|\nabla v^\epsilon|^n = O\left(\int_{|x|\geq R} |\nabla v^\epsilon|^n\right) = O\left(\int_R^\infty \frac{r^{n-1}}{r^{2n}}dr\right) = O(1). \tag{64}$$

Similarly,

$$O\left(\int_{R^n} \left(|\xi\nabla v^\epsilon|\right)^{n-1} |\nabla\xi v^\epsilon|\right) = O(1) \tag{65}$$
$$O\left(\int_{R^n} |\nabla\xi v^\epsilon|^n\right) = O(1).$$

It follows from (62)-(65)

$$\int_\Omega |\nabla (\xi v^\epsilon)| = \frac{\sqrt{n^n}\omega_n}{\epsilon^n} + O(1). \tag{66}$$

We now estimate $Q(\xi v^\epsilon)$. Applying (61) to $f(a) = v_1 \wedge \cdots \wedge v_n$ (where $a = (v^i_j)$) with $a = \xi \nabla v^\epsilon, b = \nabla \xi v^\epsilon$, we have

$$\begin{aligned} Q(\xi v^\epsilon) &= \int_\Omega \xi v^\epsilon \cdot (\xi v^\epsilon)_1 \wedge \cdots \wedge (\xi v^\epsilon)_n \\ &= \int_\Omega \xi^{n+1} v^\epsilon \cdot v^\epsilon_1 \wedge \cdots \wedge v^\epsilon_n + O\left(\int_{R^n} |\xi v^\epsilon| \left(|\xi \nabla v^\epsilon| + |\nabla \xi v^\epsilon|\right)^{n-1} |\nabla \xi v^\epsilon|\right) \end{aligned} \tag{67}$$

Recall that $Q(v^\epsilon)/(n+1)$ is the oriented volume of $v^\epsilon(R^n)$. So we have

$$Q(v^\epsilon) = \pm (n+1) \operatorname{vol}(v^\epsilon(R^n)) = \pm \frac{\omega_n}{\epsilon^{n+1}}. \tag{68}$$

Similarly to (64) and (65) , we have

$$\int_\Omega \left(\xi^{n+1} - 1\right) v^\epsilon \cdot v^\epsilon_1 \wedge \cdots \wedge v^\epsilon_n = O(1). \tag{69}$$

$$O\left(\int_{R^n} |\xi v^\epsilon| \left(|\xi \nabla v^\epsilon| + |\nabla \xi v^\epsilon|\right)^{n-1} |\nabla \xi v^\epsilon|\right) = O(1). \tag{70}$$

So (67) – (70) imply that

$$|Q(v^\epsilon)| = \frac{\omega_n}{\epsilon^{n+1}} + O(1). \tag{71}$$

Similar argument applies to $Q(\xi v^\epsilon)$, and we have

$$\begin{aligned} &\frac{1}{n+1} Q_n(\xi v^\epsilon) = \int_\Omega u \cdot (\xi v^\epsilon)_1 \wedge \cdots \wedge (\xi v^\epsilon)_n \\ &= \int_\Omega \xi^n u \cdot v^\epsilon_1 \wedge \cdots \wedge v^\epsilon_n + O\left(\int_{R^n} |u| \left(|\xi \nabla v^\epsilon| + |\nabla \xi v^\epsilon|\right)^{n-1} |\nabla \xi v^\epsilon|\right) \\ &= \int_\Omega \xi^n u \cdot v^\epsilon_1 \wedge \cdots \wedge v^\epsilon_n + O\left(1\right). \end{aligned} \tag{72}$$

Denote $\tilde{u} = \xi^n u$. Since $\tilde{u}$ is in $C^{1,\alpha}_0(B_{2R})$ by the regularity theorem in [24], we have that for all $x \in R^n$,

$$\begin{aligned}\tilde{u}(x) &= \tilde{u}(0) + \nabla\tilde{u}(0)x + O\left(\|\nabla\tilde{u}\|_{C^\alpha}|x|^{1+\alpha}\right)\\ &= u(0) + \nabla u(0)x + O\left(\|u\|_{C^{1,\alpha}(B_{2R})}|x|^{1+\alpha}\right).\end{aligned} \tag{73}$$

Therefore, by conformal invariance of Q_n and (73),

$$\begin{aligned}\int_\Omega \tilde{u}\cdot v_1^\epsilon\wedge\cdots\wedge v_n^\epsilon &= \frac{1}{\epsilon^n}\int_{R^n}\tilde{u}(\epsilon x)\cdot v_1\wedge\cdots\wedge v_n dx\\ &= \frac{1}{\epsilon^n}\left(\int_{R^n}(u(0)+\epsilon\nabla u(0)x)\cdot v_1\wedge\cdots\wedge v_n dx\right) +\\ &\quad O\left(\epsilon^{1-n+\alpha}\|\nabla u\|_{C^\alpha(B_{2R})}\int_{|x|\le\frac{2R}{\epsilon}}|x|^{1+\alpha}|\nabla v|^n\right).\end{aligned} \tag{74}$$

We have

$$\int_{R^n} u(0)\cdot v_1\wedge\cdots\wedge v_n dx = 0, \tag{75}$$

and

$$\int_{|x|\le\frac{2R}{\epsilon}}|x|^{1+\alpha}|\nabla v|^n = O(1) + \int_{1\le|r|\le\frac{2R}{\epsilon}} r^{\alpha-n} = O(1) + O(\epsilon^{n-\alpha+1}). \tag{76}$$

Next we show that

$$\int_{R^n}\nabla u(0)x\cdot v_1\wedge\cdots\wedge v_n dx = c\Sigma_{i=1}^n\frac{\partial u}{\partial x_i}\cdot e_i = c\gamma. \tag{77}$$

Proof of (77) : By (17),

$$\begin{aligned}&\int_{R^n}\nabla u(0)x\cdot v_1\wedge\cdots\wedge v_n dx\\ &= -\frac{1}{n}\int_{R^n}\sum_{i=1}^n(\nabla u(0)x)_i\cdot v_1\wedge\cdots\wedge \underset{i}{v}\wedge\cdots\wedge v_n dx\\ &= -\frac{1}{n}\int_{R^n}\sum_{i=1}^n\frac{\partial u}{\partial x_i}\cdot[e_1\wedge\cdots\wedge \underset{i}{(x,-1)}\wedge\cdots\wedge e_n]\frac{1}{\left(1+|x|^2\right)^n}dx\\ &= -\frac{1}{n}\int_{R^n}\sum_{i=1}^n\left(\frac{\partial u}{\partial x_i}\cdot e_i\right)e_i\cdot[e_1\wedge\cdots\wedge \underset{i}{(-e_{n+1})}\wedge\cdots\wedge e_n]\frac{1}{\left(1+|x|^2\right)^n}dx\\ &= c'\textstyle\sum_{i=1}^n\frac{\partial u}{\partial x_i}\cdot e_i = c'\gamma,\end{aligned}$$

where $c' = \int_{R^n} \dfrac{dx}{\left(1+|x|^2\right)^n}$. So (72) – (77) together imply that

$$\int_\Omega u \cdot (\xi v^\epsilon)_1 \wedge \cdots \wedge (\xi v^\epsilon)_n = c'\gamma\epsilon^{1-n} + O\left(\epsilon^{1-n+\epsilon}\right). \tag{78}$$

It follows from (66) (71) (440) that

$$T(\xi v^\epsilon) = \left[\epsilon^{-n}\omega_n^{\frac{n}{n+1}} + O(1)\right]^{-1} \left(\epsilon^{-n} n^{n/2}\omega_n + O(1) + \tfrac{nHc'}{n+1}\gamma + O\left(\epsilon^{-n+1+\alpha}\right)\right)$$
$$= S + c_0\epsilon + O\left(\epsilon^{1+\alpha}\right), \text{ where } c_0 = \tfrac{nHc}{n+1}\gamma < 0.$$

This finishes the proof of (60) . □

Proof of Proposition 15: Let ξv^ϵ be as in the proof Proposition 16 such that $T(\xi v^\epsilon) < S$. It is easy to check the $\pm$ sign in (68) is $(-1)^n$. Take $v = \xi v^\epsilon$ for n odd. Take $v = -\xi v^\epsilon$ for n even; so $T(v) = T(\xi v^\epsilon)$ and $Q(v) = -Q(\xi v^\epsilon)$. Thus for any n, $T(v) < S$ and $Q(v) < 0$. We may also assume that $\Phi(v) \le 0$, by replacing v by λv for large $\lambda > 0$. Consider

$$\Phi^*(tv) \equiv E_3(tv) + \frac{nH}{n+1}Q(tv) = t^n E_3(v) + \frac{nH}{n+1}t^{n+1}Q(v). \tag{79}$$

It is easy to check that Φ^* has a maximum at $t = -\frac{E_3(v)}{Q(v)H}$, with maximum value

$$\Phi^*(tv) = \left[\frac{E_3(v)}{|Q(v)|^{\frac{n}{n+1}}}\right]^{n+1} \frac{1}{|H|^n (n+1)} < \frac{S^{n+1}}{|H|^n (n+1)}. \tag{80}$$

To show (41) , we need to assume that u_0 is small, say, $\int_\Omega |\nabla u_0|^n \le 1$, then by (40)

$$\begin{aligned} |E_2(v)| &\le C \int_\Omega \sum_{i=1}^{n-1} |\nabla u_0|^{n-i} |\nabla v|^i \\ &\le C \textstyle\sum_{i=1}^{n-1} \left(\int_\Omega |\nabla u_0|^n\right)^{\frac{n-i}{n}} \left(\int_\Omega |\nabla v|^n\right)^{\frac{i}{n}} \\ &\le C \left(\int_\Omega |\nabla u_0|^n\right)^{\frac{1}{n}} \textstyle\sum_{i=1}^{n-1} \left(\int_\Omega |\nabla v|^n\right)^{\frac{i}{n}}. \end{aligned} \tag{81}$$

It follows

$$\begin{aligned}\Phi(tv) &\leq t^n E_3(v) + \frac{nH}{n+1} t^{n+1} Q(v) + C\left(\int_\Omega |\nabla u_0|^n\right)^{\frac{1}{n}} \sum_{i=1}^{n-1} t^i \left(\int_\Omega |\nabla v|^n\right)^{\frac{i}{n}} \\ &\equiv \Phi^{**}(t,v).\end{aligned} \tag{82}$$

By the construction of v, we have that

$$\begin{aligned}\epsilon^n E_3(v) = \sqrt{n^n}\omega_n + O(\epsilon);\ \epsilon^n \int_\Omega |\nabla v|^n = \sqrt{n^n}\omega_n + O(\epsilon); \\ \epsilon^{n+1} Q(v) = -\omega_n + O(\epsilon^{n+1}).\end{aligned} \tag{83}$$

Therefore, there are positive numbers C_1, C_2, C_3, such that for any number $\beta > 0$,

$$\begin{aligned}\Phi^{**}(\beta\epsilon, v) &= \beta^n \epsilon^n E_3(v) + \frac{nH}{n+1} \beta^{n+1} \epsilon^{n+1} Q(v) + \\ &C\left(\int_\Omega |\nabla u_0|^n\right)^{\frac{1}{n}} \sum_{i=1}^{n-1} \beta^i \epsilon^i \left(\int_\Omega |\nabla v|^n\right)^{\frac{i}{n}} \\ &\leq C_1 \beta^n - C_2 \beta^{n+1} + C_3 \left(\int_\Omega |\nabla u_0|^n\right)^{\frac{1}{n}} \sum_{i=1}^{n-1} \beta^i.\end{aligned} \tag{84}$$

It follows that there is a β^* such that $\Phi^{**}(\beta\epsilon, v) \leq 0$ for all $0 < \epsilon << 1$ and $\beta \geq \beta^*$. By (82) , we have,

$$\begin{aligned}\sup_{0\leq t} \Phi(tv) &\leq \sup_{0\leq t \leq \beta^*\epsilon} \Phi^{**}(t,v) \\ &\leq \sup_{0\leq t} \Phi^*(t,v) + \sup_{0\leq t\leq\beta^*\epsilon} C\left(\int_\Omega |\nabla u_0|^n\right)^{\frac{1}{n}} \sum_{i=1}^{n-1} t^i \left(\int_\Omega |\nabla v|^n\right)^{\frac{i}{n}} \\ &\leq \sup_{0\leq t} \Phi^*(t,v) + C\left(\int_\Omega |\nabla u_0|^n\right)^{\frac{1}{n}} \sum_{i=1}^{n-1} \beta^{*i} \epsilon^i \left(\int_\Omega |\nabla v|^n\right)^{\frac{i}{n}} \\ &\leq \sup_{0\leq t} \Phi^*(t,v) + C_4 \left(\int_\Omega |\nabla u_0|^n\right)^{\frac{1}{n}},\end{aligned} \tag{85}$$

where C_4 depends on β^*. Since $\sup_{0\leq t} \Phi^*(t,v) < \dfrac{S^{n+1}}{H^n\,(n+1)}$, $\sup_{0\leq t} \Phi(tv) < \dfrac{S^{n+1}}{H^n\,(n+1)}$, if $\int_\Omega |\nabla u_0|^n$ is small enough. □

4 Regularity of Conformal Solutions

Our result is

Theorem 17 *If u is a conformal solution of* (1) *and f satisfies* (2)*, then $u \in C^{1,\alpha}(\Omega)$ for some $\alpha \in (0,1)$. If $u = \eta$ on $\partial\Omega$ and $\eta \in C^0(\partial\Omega)$, then $u \in C^0(\bar{\Omega})$.*

When $n = 2$, this theorem was proved by Grüter in 1980 [16]. We will use the main idea of the proof in [16].

Consider the set G of *good* points of u defined by

$$G = \{x \in \Omega : \begin{array}{l} u \text{ is } \textit{approximately differentiable} \text{ at } x, \text{and} \\ x \text{ is a Lebesgue point of } |\nabla u|^n, \text{ and } |\nabla u|(x) \neq 0\}. \end{array}$$

Here u is *approximately differentiable* at a point x_0 with approximate *differential* $\nabla u(x_0)$, by definition, if there is a $u_0 \in R^k$ such that for every $\epsilon > 0$,

$$\Phi^n[L^n\lfloor\Omega\backslash\{x : |u(x) - u_0 - \nabla u(x_0)(x - x_0)| \leq \epsilon|x - x_0|\}, x_0] = 0,$$

where Φ^n denotes the n-dimensional *density* and $L^n\lfloor\Omega$ is the Lebesgue measure, restricted to Ω.

We will need the following property for functions in $W^{1,n}(\Omega, R^k)$.

Proposition 18 *([12] [Theorem 4.5.9]) If $u \in W^{1,n}(\Omega, R^k)$, then u has weak derivative and approximate differential almost everywhere, and when both exist, they coincide.*

For a proof, see [12]. Next, we have

Lemma 19 *Suppose $u \in W^{1,n}(\Omega, R^k)$ and $B \subset \Omega$ is a ball. Then*

$$\operatorname{osc}_B u \leq 4max\{\alpha_1, \alpha_2\}, \tag{86}$$

where $\alpha_1 = osc_{\partial B}\, u$, $\alpha_2 = \underset{y \in G\cap B}{sup} \ \underset{x \in \partial B}{inf} |u(x) - u(y)|$.

Proof: The proof is similar to that in [16]. Denote α =max$\{\alpha_1, \alpha_2\}$. Take a point $x_1 \in \partial B$. Define $z = u - u(x_1)$ and v =max$\{|z| - 2\alpha, 0\}$. Then $v \in W_0^{1,n}(\Omega)$. From the definitions of α_1,α_2, one sees that $v = 0$ on $G \cap B$. It follows that $\nabla v(y) = 0$ if $y \in G \cap B$. On the complement of $G \cap B$, $\nabla v = 0$ almost everywhere. Therefore $\nabla v = 0$ almost everywhere on B, and so v must be a constant, which is zero. □

We also need the Courant-Lebesgue Lemma.

Lemma 20 *Suppose $u \in W^{1,n}\left(\Omega, R^k\right)$ and $B(x,r) \subset \Omega$, $0 < r < 1$. Then there is a constant $C > 0$ and some $\delta \in [\frac{r}{2}, r]$ such that*

$$\underset{\partial B(x,\delta)}{osc}\, u \le CK^{1/n}, \quad \text{where } K = \int_{B(x,r)} |\nabla u|^n. \tag{87}$$

Proof: Recall that for $y \in B(x,r)$, $|\nabla u(y)|^2 \ge \rho^{-2}|\nabla_\theta u(y)|^2$, where $\rho = |y - x|$ and $\theta = \dfrac{y-x}{\rho} \in S^{n-1}$. It follows

$$\int_{\frac{r}{2}}^{r} \int_{S^{n-1}} \rho^{-1} |\nabla_\theta u(y)|^n d\theta d\rho \le K. \tag{88}$$

By Fubini's theorem, there is a $\delta \in [\frac{r}{2}, r]$ such that

$$\int_{\frac{r}{2}}^{r} \int_{S^{n-1}} \delta^{-1} |\nabla_\theta u(y)|^n d\theta d\rho = \frac{r}{2\delta} \int_{S^{n-1}} |\nabla_\theta u(y)|^n d\theta. \tag{89}$$

Since $\delta \le r$, (88) (89) imply that $\int_{S^{n-1}} |\nabla_\theta u(y)|^n d\theta \le 2K$. (87) follows from Sobolev embedding theorem $W^{1,n}(S^{n-1}, R^k) \hookrightarrow C^{1/n}$. □

This lemma gives a control of the oscillation of u on the boundary $\partial B(x,\delta)$. Our following step is to estimate the interior oscillation. We need some propositions.

Proposition 21 *Suppose $u \in W^{1,n}(\Omega, R^k)$ is conformal and $B \subset \Omega$ is an open subset. Define $D_\sigma = B \cap \{x : |u(x) - u(x_0)| < \sigma\}$ for $x_0 \in B \cap G$ and $\sigma > 0$. Then*

$$\underset{\sigma\to 0}{limsup}\ \sigma^{-n} \int_{D_\sigma} |\nabla u|^n \ge n^{\frac{n}{2}-1} \omega_{n-1}, \tag{90}$$

where ω_{n-1} is the area of the sphere S^{n-1}.

Remark 22 *This Proposition was proved by Grüter [16] when* $n = 2$.

Remark 23 *Without the assumption that* u *is conformal, then* (90) *still holds, with the right hand being replaced by* $n^{-1}\omega_{n-1}$.

Proof: We may assume $u(x_0) = 0$. For $\epsilon, \sigma > 0$, define

$$T_\epsilon = B \backslash \{x : |u(x) - \nabla u(x_0)(x - x_0)| \leq \frac{\epsilon}{\sqrt{n}}|x - x_0|\},$$
$$B_\epsilon = B \cap \{x : |x - x_0| < r_\epsilon\}, \text{ where } r_\epsilon = \frac{\sigma\sqrt{n}}{|\nabla u(x_0)| + \epsilon}.$$

We claim

$$B_\epsilon \backslash T_\epsilon \subset D_\sigma \backslash T_\epsilon.$$

Indeed, if $x \in B_\epsilon \backslash T_\epsilon$, then

$$|u(x) - \nabla u(x_0)(x - x_0)| \leq \frac{\epsilon}{\sqrt{n}}|x - x_0|;$$

while the conformality condition (8) implies that

$$|\nabla u(x_0)(x - x_0)|^2 \leq \frac{1}{n}|\nabla u(x_0)|^2 |x - x_0|^2.$$

Therefore,

$$|u(x)| \leq \frac{1}{\sqrt{n}}(|\nabla u(x_0)| + \epsilon)|x - x_0| < \sigma,$$

and so $x \in D_\sigma \backslash T_\epsilon$.

Note that for any $a, b \geq 0, \epsilon > 0$,and $p > 1$, there holds

$$a^p \geq (1 - \epsilon)b^\epsilon - (\epsilon^{-1} - 1)|a^p - b^p|.$$

(For a proof, note that $\epsilon b^p + \epsilon^{-1}|a^p - b^p| \geq 2b^{p/2}|a^p - b^p|^{1/2} \geq |a^p - b^p| - (a^p - b^p)$.) Using this inequality, we obtain

$$\begin{aligned} \sigma^{-n}\int_{D_\sigma} |\nabla u(x)|^n &\geq \sigma^{-n}\int_{B_\epsilon \backslash T_\epsilon} |\nabla u(x)|^n \\ &\geq \sigma^{-n}(1 - \epsilon)\int_{B_\epsilon \backslash T_\epsilon} |\nabla u(x_0)|^n - \\ &\quad -\sigma^{-n}(\epsilon^{-1} - 1)\int_{B_\epsilon \backslash T_\epsilon} (|\nabla u(x)|^n - |\nabla u(x_0)|^n). \end{aligned} \tag{91}$$

We look at each term in (91) as $\sigma \to 0$. For the first term,

$$\begin{aligned} & \sigma^{-n}(1-\epsilon)\int_{B_\epsilon\backslash T_\epsilon} |\nabla u(x_0)|^n = \sigma^{-n}(1-\epsilon)|\nabla u(x_0)|^n L^n(B_\epsilon\backslash T_\epsilon) \qquad (92) \\ \to \quad & \frac{\omega_{n-1}}{n}\sqrt{n^n}(1-\epsilon)\left(\frac{|\nabla u(x_0)|}{|\nabla u(x_0)|+\epsilon}\right)^n (1-\Phi^n(L\lfloor T_\epsilon, x_0)) \\ = \quad & \tfrac{\omega_{n-1}}{n}\sqrt{n^n}(1-\epsilon)\left(\frac{|\nabla u(x_0)|}{|\nabla u(x_0)|+\epsilon}\right)^n. \end{aligned}$$

For the second term in (91), using that x_0 is a Lebesgue point of $|\nabla u|^n$, we have, for fixed ϵ,

$$\begin{aligned} & \sigma^{-n}(\epsilon^{-1}-1)\int_{B_\epsilon\backslash T_\epsilon} ||\nabla u(x)|^n - |\nabla u(x_0)|^n)| \\ & \le \frac{(\epsilon^{-1}-1)\sqrt{n^n}}{(|\nabla u(x_0)|+\epsilon)^n L^n(B_\epsilon)}\int_{B_\epsilon\backslash T_\epsilon} ||\nabla u(x)|^n - |\nabla u(x_0)|^n| \qquad (93) \\ & \to 0 \text{ as } \sigma \to 0. \end{aligned}$$

Now taking the limit in (91) and using (92) (93), we obtain (90). □

Theorem 24 *Suppose $B \subset \Omega$ is a ball, $x_0 \in B \cap G$, and $\Sigma > 0$ is a number such that*

$$2n\Lambda\Sigma \le 1, \qquad (94)$$

$$dist\,(u(\partial B), u(x_0)) > \Sigma, \qquad (95)$$

where Λ is as in (2). Then

$$\int_B |\nabla u|^n \ge \frac{1}{2}\omega_{n-1} n^{\frac{n}{2}-1}\Sigma^n. \qquad (96)$$

Corollary 25 *Suppose $B \subset \Omega$ is a ball such that*

$$\int_B |\nabla u|^n \le \frac{\omega_{n-1}}{2^{\bar{n}+2} n^{n/2+1}\Lambda^n}.$$

Then for any $x_0 \in B \cap G$,

$$dist\;(u(\partial B), u(x_0)) \le \frac{1}{\omega_{n-1} n^{\frac{n}{2}-1}}\left(\int_B |\nabla u|^n\right)^{1/n}.$$

Proof of Corollary 25: Let

$$\Sigma = \left(\frac{4}{\omega_{n-1} n^{n/2-1}} \int_B |\nabla u|^n \right)^{1/n}.$$

Then the condition (94) is satisfied, but the conclusion (96) does not hold, therefore, (95) must not hold. □

Proof of Theorem 24: For $\sigma \in (0, \Sigma]$, denote

$$D_\sigma = B \cap \{x : |u(x) - u(x_0)| < \sigma\}.$$

Let $\lambda \in C_0^1(R, [0,1]\,)$ be a function such that $\lambda(t) = 0$ for $t \leq 0$. For $\rho \in (0, \Sigma)$, define

$$\eta = \lambda(\rho - |u|)u.$$

From (94), we have $\eta \in W_0^{1,n}(B, R^k) \cap L^\infty$. Multiplying η to the equation (1) and integrating by parts, we obtain

$$\begin{aligned} \int_{D_\rho} |\nabla u|^n \lambda(\rho - |u|) - \int_{D_\rho} |\nabla u|^{n-2} \lambda'(\rho - |u|) |\nabla u \cdot \frac{u}{|u|}|^2 \\ = \int_{D_\rho} f(x, u, \nabla u)\, u \lambda(\rho - |u|). \end{aligned} \tag{97}$$

Define

$$\Phi(\rho) = \frac{1}{n} \int_{D_\rho} |\nabla u|^n \lambda(\rho - |u|).$$

Then we have

$$\Phi'(\rho) \geq \frac{1}{n} \int_{D_\rho} |\nabla u|^n \lambda'(\rho - |u|). \tag{98}$$

From the conformality of u, it follows that $|\nabla u \cdot u|^2 \leq \frac{1}{n} |\nabla u|^2 |u|^2$. The property of λ implies that

$$\lambda'(\rho - |u|)\, |u| \leq \rho \lambda'(\rho - |u|)\,.$$

Therefore, we have that

$$\begin{aligned} \int_{D_\rho} |\nabla u|^{n-2} \lambda'(\rho - |u|) |\nabla u \cdot \frac{u}{|u|}|^2 \leq \frac{1}{n} \int_{D_\rho} |\nabla u|^n \lambda'(\rho - |u|)\, |u| \\ \leq \frac{1}{n} \rho \int_{D_\rho} |\nabla u|^n \lambda'(\rho - |u|) \leq \rho \Phi'(\rho). \end{aligned} \tag{99}$$

Also, we have

$$\begin{aligned}\int_{D_\rho} f(x,u,\nabla u)\, u\lambda(\rho-|u|) &\le \Lambda \int_{D_\rho} |\nabla u|^n \lambda(\rho-|u|)\,|u| \\ &\le \Lambda \int_{D_\rho} |\nabla u|^n \int_0^\rho \lambda'(\sigma-|u|)|u|d\sigma \\ &\le \Lambda \int_0^\rho \sigma \left(\int_{D_\rho} |\nabla u|^n \lambda'(\sigma-|u|)|u|\right) d\sigma \\ &\le n\Lambda \int_0^\rho \sigma\Phi'(\sigma)d\sigma.\end{aligned} \tag{100}$$

Thus (97) together with (99) and (100) yields

$$n\Phi(\rho)-\rho\Phi'(\rho) \le n\Lambda \int_0^\rho \sigma\Phi'(\sigma)d\sigma. \tag{101}$$

This can be rewritten as

$$-\left(\frac{\Phi(\rho)}{\rho^n}\right)' \le \frac{n\Lambda}{\rho^{n+1}} \int_0^\rho \sigma\Phi'(\sigma)d\sigma \le n\Lambda\frac{\Phi(\rho)}{\rho^n}. \tag{102}$$

This differential inequality implies that $e^{n\Lambda\rho}\dfrac{\Phi(\rho)}{\rho^n}$ is increasing in ρ; in particular, $\dfrac{\Phi(\rho)}{\rho^n}$ has a limit as $\rho \to 0$. Furthermore, for $0 < \rho_1 \le \rho_2 \le \Sigma$, by integrating 102 from ρ_1 to ρ_2, we have

$$\frac{\Phi(\rho_1)}{\rho_1^n} \le \frac{\Phi(\rho_2)}{\rho_2^n} + n\Lambda \int_0^{\rho_2} \frac{\Phi(\rho)}{\rho^n} d\rho. \tag{103}$$

The second term of (103) can be estimated by integration by parts and using (101)

$$\begin{aligned}\int_0^{\rho_2} \frac{\Phi(\rho)}{\rho^n} d\rho &\le \frac{\Phi(\rho_2)}{\rho_2^{n-1}} + \int_0^{\rho_2} \rho\left(-\frac{\Phi(\rho)}{\rho^n}\right)' d\rho \\ &\le \frac{\Phi(\rho_2)}{\rho_2^{n-1}} + n\Lambda \int_0^{\rho_2} \frac{1}{\rho^n} \int_0^\rho \sigma\Phi'(\sigma)d\sigma d\rho \\ &\le \frac{\Phi(\rho_2)}{\rho_2^{n-1}} + n\Lambda \int_0^{\rho_2} \frac{\Phi(\rho)}{\rho^{n-1}} d\rho \\ &\le \frac{\Phi(\rho_2)}{\rho_2^{n-1}} + n\Lambda\rho_2 \int_0^{\rho_2} \frac{\Phi(\rho)}{\rho^n} d\rho.\end{aligned} \tag{104}$$

From the assumption, $n\Lambda\rho_2 \le n\Lambda\Sigma \le \frac{1}{2}$. Thus it follows that from (104)

$$\int_0^{\rho_2} \frac{\Phi(\rho)}{\rho^n} d\rho \le 2\frac{\Phi(\rho_2)}{\rho_2^{n-1}}. \tag{105}$$

Now (103) and (105) imply that

$$\frac{\Phi(\rho_1)}{\rho_1^n} \le \frac{\Phi(\rho_2)}{\rho_2^n} + 2n\Lambda\frac{\Phi(\rho_2)}{\rho_2^{n-1}} \le 2\frac{\Phi(\rho_2)}{\rho_2^n}. \tag{106}$$

Given $\epsilon > 0$, we choose $\lambda(t)$ with the additional property that $\lambda(t) = 1$ for $t \ge \epsilon\rho_{1.}$ Then it easy to see

$$\frac{1}{n}\int_{D_{\rho_1(1-\epsilon)}} |\nabla u|^n \le \Phi(\rho_1), \quad \Phi(\rho_2) \le \frac{1}{n}\int_{D_{\rho_2}} |\nabla u|^n. \tag{107}$$

Apply (106) with $\rho_2 = \Sigma$, and use (107), we then obtain

$$\frac{\int_{D_{\rho_1(1-\epsilon)}} |\nabla u|^n}{\rho_1^n} \le 2\frac{\int_{D_\Sigma} |\nabla u|^n}{\Sigma^n} \le 2\frac{\int_B |\nabla u|^n}{\Sigma^n}. \tag{108}$$

Let $\rho_1 \to 0$ in (108) and apply Proposition 21, and then send $\epsilon \to 0$. (96) then follows. □

Proof of Theorem 17. We divide the proof into several steps. The first step, showing the continuity of u, is the essential one. The other steps are standard.

(a). *u is continuous.* Fix $x_0 \in \Omega$. Choose $R > 0$ small enough such that $B(x_0,R) \subset \Omega$ and $\int_{B(x_0,R)} |\nabla u|^n \le \frac{\omega_{n-1}}{2^{\bar{n}+2} n^{n/2+1}\Lambda^n}$. By the Courant-Lebesgue Lemma 20, there is a $\delta \in [\frac{R}{2}, R]$ such that

$$\operatorname*{osc}_{\partial B(x_0,\delta)} u \le C\left(\int_{B(x_0,R)} |\nabla u|^n\right)^{1/n} \equiv \alpha_1(R).$$

Because $\int_{B(x_0,\delta)} |\nabla u|^n \le \int_{B(x_0,R)} |\nabla u|^n$, by Corollary 25, for any $x_0 \in B \cap G, B = B(x_0, \delta)$,

$$\text{dist } (u(\partial B), u(x_0)) \le \frac{1}{\omega_{n-1} n^{\frac{n}{2}-1}}\left(\int_{B(x_0,R)} |\nabla u|^n\right)^{1/n} \equiv \alpha_2(R).$$

Now Lemma 19 implies that

$$\operatorname{osc}_{B(x_0,\delta)} u \leq 4\max\{\alpha_1(R), \alpha_2(R)\} \to 0 \text{ as } R \to 0. \tag{109}$$

In particular, $\operatorname{osc}_{B(x_0,R/2)} u \to 0$ as $R \to 0$. So u is continuous at x_0.

(b). *u is C^β for some $\beta \in (0,1)$.* We claim that if $x_0 \in W$ and $R > 0$ such that $B(x_0,R) \subset \Omega$ and $osc_{B(x_0,R)} u\,\Lambda < 1$, then there is a number $\tau \in (0,1)$ so that for every ball $B(x,r) \subset B(x_0,R)$

$$\int_{B(x,r/2)} |\nabla u|^n \leq \tau \int_{B(x,r)} |\nabla u|^n. \tag{110}$$

By iteration, we have for some constants C and $\gamma \in (0,1)$,

$$\int_{B(x,r)} |\nabla u|^n \leq C r^\gamma \text{ for } x \in B(x_0,R/2) \text{ and } r \in (0,R/2).$$

That u is C^β for some $\beta \in (0,1)$ follows from Morrey's Lemma [23][3.5.2]. The proof of (110) is a standard "hole-filling" method. Let $\bar{u}$ be the mean value of u on $B(x,r)\backslash B(x,r/2)$ and $\eta \in C_0^1(B(x,r),[0,1])$ be a cut-off function such that $\eta = 1$ on $B(x,r/2)$ and $|\nabla\eta| \leq 3/r$. Take $\phi = (u-\bar{u})\eta$, then $\phi \in W_0^{1,n}(B(x,r),R^k)$. Multiply ϕ to the equation (1) and integrate. We then get

$$\begin{aligned} & \left|\int_{B(x,r)} \eta|\nabla u|^n + \int_{B(x,r)} |\nabla u|^{n-2}\nabla u\nabla\eta(u-\bar{u})\right| \\ = & \left|\int_{B(x,r)} f(x,u,\nabla u)\eta(u-\bar{u})\right| \\ \leq & \; osc_{B(x_0,R)} u\,\Lambda \int_{B(x,r)} |\nabla u|^n. \end{aligned} \tag{111}$$

By Hölder and Poincare's inequalities, we estimate the second term of (111),

$$\begin{aligned} & \left|\int_{B(x,r)} |\nabla u|^{n-2}\nabla u\nabla\eta(u-\bar{u})\right| \\ \leq & \; C_1\left(\int_{B(x,r)\backslash B(x,r/2)} |\nabla u|^n\right)^{(n-1)/n} \left(\frac{1}{r}\int_{B(x,r)\backslash B(x,r/2)} |u-\bar{u}|^n\right)^{1/n} \\ \leq & \; C_2 \int_{B(x,r)\backslash B(x,r/2)} |\nabla u|^n, \end{aligned} \tag{112}$$

where C_1 and C_2 depend only on n and k. Put (112) back to (111) and note the property of η. It follows

$$\int_{B(x,r/2)} |\nabla u|^n - C_2 \int_{B(x,r)\setminus B(x,r/2)} |\nabla u|^n \leq osc_{B(x_0,R)} u\, \Lambda \int_{B(x,r)} |\nabla u|^n$$

or

$$(C_2+1) \int_{B(x,r/2)} |\nabla u|^n \leq \left(C_2 + osc_{B(x_0,R)} u\, \Lambda\right) \int_{B(x,r)} |\nabla u|^n$$

Let $\tau = \left(C_2 + osc_{B(x_0,R)} u\, \Lambda\right) / (C_2+1)$. Then (43) follows.

(c) . *u is $C^{1,\alpha}$ for some $\alpha \in (0,1)$* . For the proof of $C^{1,\alpha}$ regularity based on C^β, we refer [17] or [13].

(d). *u is continuous up to $\partial\Omega$*. This was proved in [24] [Theorem 4.1]. □

Acknowledgement. The authors wish to thank M. Fuchs for bring up their attention to his results in [10][14][9] and other works on surfaces with prescribed mean curvature vectors and on n-harmonic maps.

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