Regularity for n-Harmonic Maps

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Abstract. Here we obtain everywhere regularity of weak solutions of some nonlinear elliptic systems with borderline growth, including n-harmonic maps between manifolds or map with constant volumes. Other results in this paper include regularity up to the boundary and a removability theorem for isolated singularities.

§ 1. Introduction

Let n, $m \geq 2$ be integers, $p \in (1, n]$ and Ω be a smooth bounded domain $\Omega \subset \mathbf{R}^n$. As usual, $W^{1,p}(\Omega, \mathbf{R}^m)$ is the set of all functions $u \in L^p(\Omega, \mathbf{R}^m)$ with finite p-energy $\int_\Omega |\nabla u|^p < \infty$; it is a Banach space with the norm $\|u\|_{W^{1,p}} = \int_\Omega |u|^p + |\nabla u|^p$.

Let K be a submanifold of $W^{1,p}(\Omega, \mathbf{R}^m)$. A *p-harmonic map* in K is a critical point in K of the energy functional $\int_\Omega |\nabla u|^p$. Denote by T_uK the tangent space of K at u, that is, the set of all $\phi \in W^{1,p}(\Omega, \mathbf{R}^m)$, where $\phi = \frac{d}{dt}\big|_{t=0} u_t$ for a smooth curve u_t in K with $u_0 = u$. A p-harmonic map in K thus satisfies

$$\operatorname{div}(|\nabla u|^{p-2}\nabla u) \perp T_uK.$$

This relation can often be written as an equation:

$$\operatorname{div}(|\nabla u|^{p-2}\nabla u) = f(x, u, \nabla u), \tag{1.1}$$

where f is smooth and homogeneous in ∇u of degree p. We now look at two examples.

First, suppose that N is a compact Riemannian manifold, which we may assume is isometrically embedded into $\mathbf{R}^m$ as a submanifold. Let K be the submanifold consisting of all $u \in W^{1,p}(\Omega, \mathbf{R}^m)$ with image $u(\Omega) \subset N$ and with fixed boundary data $u|\partial\Omega$. Then p-harmonic maps in K are called *p-harmonic maps* from Ω to N, and (1.1) becomes

$$\operatorname{div}(|\nabla u|^{p-2}\nabla u) - |\nabla u|^{p-2} A(u)(\nabla u, \nabla u) = 0, \tag{1.2}$$

where $A(u)$ is the second fundamental form of N. For a derivation, see [SU1] and [HL1]. In particular, when $N = \mathbf{S}^{m-1}$, (1.2) reduces to

$$\operatorname{div}(|\nabla u|^{p-2}\nabla u) + |\nabla u|^p u = 0. \tag{1.3}$$

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The second example occurs when $m = n+1$ and $p = n$. In this case, a function $u \in W^{1,n}(\Omega, \mathbf{R}^m)$ parametrizes a *generalized* hypersurface $u(\Omega) \subset \mathbf{R}^m$. If $u \in C^1(\bar{\Omega}, \mathbf{R}^m)$, then the cone generated by $u(\Omega)$ (with vertex $0 \in \mathbf{R}^m$) has a oriented volume

$$\mathcal{V}(u) = \frac{1}{n+1} \int_\Omega u \cdot \mathbf{J}(u), \tag{1.4}$$

where $\mathbf{J}(u) = \partial_1 u \wedge \ldots \wedge \partial_n u$ is the wedge product of $\partial_1 u, \ldots, \partial_n u$; it is the vector whose components are the $n \times n$ minors of the Jacobi matrix $\left(\partial_j u^i\right)_{(n+1)\times n}$. We will see that $\mathcal{V}(u)$ can be defined by (1.4) for any $u \in W^{1,n}(\Omega, \mathbf{R}^m)$ with bounded $u|\partial\Omega$. Let K be the set of all $u \in W^{1,n}(\Omega, \mathbf{R}^m)$ with prescribed $u|\partial\Omega$ and $\mathcal{V}(u)$. We call the critical points $u \in K$ of $\int_\Omega |\nabla u|^n$ *n-harmonic maps with constant volume.* These satisfy

$$\operatorname{div}(|\nabla u|^{n-2}\nabla u) + H\ \mathbf{J}(u) = 0, \tag{1.5}$$

for some constant H.

By a *p-harmonic map*, we will simply mean a weak solution of (1.1). Here we are concerned with the regularity of p-harmonic maps, especially n-harmonic maps into spheres or maps with constant volumes.

Theorem 2.6 asserts that if u is a p-harmonic map with the monotonicity property and $f(x, u, \nabla u)$ is in the local Hardy space $\mathcal{H}^1_{\text{loc}}(\Omega)$ (a space slightly smaller than L^1), then u is partially regular to the extent that the singular set has 0 Hausdorff measure of dimension $n-p$. Moreover, we show that n-harmonic maps are everywhere regular in the interior, continuous up to the boundary of Ω, and have removable isolated singularities. See Theorems 3.2, 3.6, 4.1 and 5.1.

The regularity of 2-harmonic maps (on a disk) with constant volume and H-surfaces was proved by Wente [WH], Grüter [G], and others. In this case, any solution of $\Delta u = H\partial_1 \wedge \partial_2 u$ is analytic [WH, 1969], in contrast with our optimal $C^{1,\alpha}$ regularity for some $\alpha \in (0,1)$ (see [L]).

Partial regularity of harmonic maps to manifolds has been extensively studied. For minimizers of $\int_\Omega |\nabla u|^p$, see the work of Schoen and Uhlenbeck, Hardt and Lin, and Luckhaus [SU1][HL1][LS]. For stationary 2-harmonic maps to spheres, see Evans' paper [EL]. Theorem 3.2 generalizes Evans' result to p-harmonic maps. Very recently, Bethuel [B] proved partial regularity for 2-harmonic maps with monotonicity for an arbitrary target manifold.

A 2-harmonic map on a surface is regular up to the boundary, see Heléin [HF1,2] and Qing [QJ]. For earlier results in this case, see [MC], [SR].

Removability of isolated singularities of 2-harmonic maps on a surface was proved by Sacks and Uhlenbeck [SaU, Thm 3.6], who also applied the result to find harmonic maps in homotopy classes. Theorem 5.1 generalizes this result to arbitrary higher dimensional cases, which can be applied to show the existence of n-harmonic maps homotopic to given maps.

Our proof of Theorem 2.6 uses a blow-up argument, which leads to an energy growth estimate. We prove the strong convergence of the blow-up sequence by using Fefferman-Stein's duality theorem

$\mathcal{H}^1(\mathbf{R}^n)^* = \mathrm{BMO}(\mathbf{R}^n)$ and the assumption that f is in the Hardy space $\mathcal{H}^1$. To apply Theorem 2.6 to (1.3) and (1.5), we verify that $f \in \mathcal{H}^1_{\mathrm{loc}}$ by using a criterion in [CLMS2].

In the proof of Theorem 5.1 we used the idea in [SaU]. Assuming that $f \perp \nabla u$ and $u \in C^1$, we show that u satisfies a monotonicity identity (Corollary 2.2), which is essential to the proof. In fact, the result is not true without this assumption; for examples, see [G].

The monotonicity property (defined in Corollary 2.2) in Theorem 2.6 is also essential when $p \in (1, n)$. Riviere constructed an example [RT] of 2-harmonic map from $\mathbf{B}^3$ to $\mathbf{S}^2$ that is everywhere discontinuous and has no monotonicity property. For more discussions on the monotonicity, see [SU1][HL1][SR][DF][CL][ML].

We now make some remarks on the notations. We use the summation convention that repeated indices are summed over. $\frac{\partial u}{\partial x_\alpha}$ is also denoted by $\partial_\alpha u$ or u_α. For $r > 0$, we denote by $\mathbf{B}_r$ a ball of radius r, and $\mathbf{S}_r = \partial \mathbf{B}_r$. To indicate the center of the ball, say x, we write $\mathbf{B}_r(x)$, and $\mathbf{S}_r(x) = \partial \mathbf{B}_r(x)$. The integral measures are suppressed when they are clear from the domain, for example, $\int_\Omega f(x) = \int_\Omega f(x)dx$, $\int_{\partial \mathbf{B}_r} f(x) = \int_{\partial \mathbf{B}_r} f(x) d\mathbf{S}_r$. The constants C may vary from line to line; their dependence other than n, m, p will be described or indicated by a $C(\delta, M)$, etc.

We thank Tom Wolff for bringing to our attention the article of Lewis. Recently we learned that Hardt and Lin [HL2] also had a proof for Theorem 3.2 in this paper.

§ 2. Interior regularity theorem

In this section we consider the regularity of solutions of

$$\operatorname{div}(|\nabla u|^{p-2}\nabla u) = f(x, u, \nabla u), \tag{2.1}$$

where f is assumed (through this paper) to be smooth, and there is a non-decreasing function $\mu : [0, \infty) \to [0, \infty)$ so that for $x \in \bar{\Omega}$, $y \in \mathbf{R}^m$ and $z \in \mathbf{R}^{mn}$,

$$|f|, |f_x|, |f_y|, |z||f_z| \le \mu(|y|)|z|^p < \infty. \tag{2.2}$$

Sometimes we also assume that

$$f(x, u, \nabla u) \perp \nabla u, \tag{2.3}$$

that is, $f \perp \partial_\alpha u$ for each α. Note that the both equations (1.2) and (1.5) satisfy (2.3).

Theorem 2.1. *Suppose that $u \in C^1(\Omega, \mathbf{R}^m)$ is a weak solution of (2.1) such that u and f satisfy (2.2) and (2.3). Then u satisfies the monotonicity identity:*

$$\frac{d}{dr}\left(r^{p-n}\int_{\mathbf{B}_r}|\nabla u|^p\right) = pr^{p-n}\int_{\partial \mathbf{B}_r}|\nabla u|^{p-2}|u_r|^2 \tag{2.4}$$

for any $\mathbf{B}_r = \mathbf{B}_r(x_0) \subset \Omega$. Here $r = |x - x_0|$ and $u_r = \frac{\partial u}{\partial r}$.

Proof. The monotonicity identity can be derived from the following identity:

$$\int_\Omega |\nabla u|^p \mathrm{div} X - p|\nabla u|^{p-2}\partial_\alpha u \partial_\beta u \partial_\alpha X^\beta = 0, \tag{2.5}$$

for $X \in C_0^1(\Omega, \mathbf{R}^n)$, as in [DF][ML].

To verify (2.5), let $X \in C_0^1(\Omega, \mathbf{R}^n)$. If X is supported in $\Omega_+ = \{x \in \Omega : |\nabla u|(x) > 0\}$ where u is actually C^2, then we may take $\phi \equiv \sum_\alpha \frac{\partial u}{\partial x_\alpha} X^\alpha \in C_0^1(\Omega_+, \mathbf{R}^n)$. Then by (2.3) we have $\phi \cdot f(x, u(x), \nabla u(x)) = 0$. Thus (2.1) yields

$$\begin{aligned} 0 &= \int_\Omega |\nabla u|^{p-2} \nabla u \nabla \phi \\ &= \int_\Omega \frac{1}{p} \partial_\alpha (|\nabla u|^p) X^\alpha + |\nabla u|^{p-2} \partial_\alpha u \partial_\beta u \partial_\alpha X^\beta \\ &= \int_\Omega -\frac{1}{p} |\nabla u|^p \mathrm{div} X + |\nabla u|^{p-2} \partial_\alpha u \partial_\beta u \partial_\alpha X^\beta. \end{aligned}$$

So (2.5) holds for $X \in C_0^1(\Omega_+, \mathbf{R}^n)$. Since $\nabla u(x) = 0$ on $\Omega \setminus \Omega_+$, (2.5) is true for all $X \in C_0^1(\mathbf{B}_1 \setminus \{0\}, \mathbf{R}^n)$ as shown by an approximation in [DF]. ■

For stationary harmonic maps to manifolds, (2.4) was showed in [HL1, p570].

Corollary 2.2. *If u is as in Theorem 2.1, then u has monotonicity property, that is, the normalized energy $r^{p-n} \int_{\mathbf{B}_r(x)} |\nabla u|^p$ is increasing in r, where $\mathbf{B}_r(x) \subset \Omega$. In particular, if $p = n$, then*

$$\int_{\partial \mathbf{B}_r} |\nabla u|^n = n \int_{\partial \mathbf{B}_r} |\nabla u|^{n-2} |u_r|^2. \tag{2.6}$$

Proof. Monotonicity of u and (2.6) directly follow from previous theorem. ■

Remark. If fact (2.4) and (2.6) hold for $\mathbf{B}_r = \mathbf{B}_r(x_0)$ even if we only assume $u \in C^1(\Omega \setminus \{x_0\}, \mathbf{R}^m)$. In this case (2.5) holds for $X \in C_0^1(\Omega \setminus \{x_0\}, \mathbf{R}^n)$.

Theorem 2.3.

(a). *If $u \in W^{1,p}(\Omega, \mathbf{R}^m) \cap C^{0,\beta}(\Omega, \mathbf{R}^m)$ for some $\beta \in (0,1)$ is a solution of (2.1), then $u \in C^{1,\alpha}(\Omega, \mathbf{R}^m)$ for some $\alpha \in (0,1)$.*

(b). *There are positive numbers ε and C, depending on $p, n, m, \mu(M)$, so that if $u \in C^1(\Omega, \mathbf{R}^m)$ is a solution of (2.1) with $|u| \le M$ and if u satisfies (2.3) and $r^{p-n} \int_{\mathbf{B}_r(x)} |\nabla u|^p \le \varepsilon$, then*

$$\sup_{\mathbf{B}_{\frac{r}{2}}(x)} |\nabla u|^p \le C \fint_{\mathbf{B}_r(x)} |\nabla u|^p.$$

Proof. (a) can be proved by following the outline of the proof of Theorem 3.1 in [HL1].

(b) was proved for p-harmonic maps to manifolds in [DF, Thm 2.1]. The main ingredient of the proof was monotonicity property, which in our case follows from Corollary 2.2. ■

When $f \equiv 0$, the solutions of (2.1) are p-harmonic functions, whose regularity is well-known, proved by Uhlenbeck [UK, Thm. 3.2 & 5.4] for $p \geq 2$ and Tolksdorff [T] for $p > 1$:

Theorem 2.4. *If $u \in W^{1,p}(\mathbf{B}_r(x), \mathbf{R}^m)$ is p-harmonic, then $u \in C^{1,\alpha}(\mathbf{B}_r(x), \mathbf{R}^m)$ for some $\alpha \in (0,1)$, and for some constant $C(n,m,p)$,*

$$\sup_{\mathbf{B}_{\frac{r}{2}}(x)} |\nabla u|^p \leq C \fint_{\mathbf{B}_r(x)} |\nabla u|^p.$$

We will use a simple inequality: For an integer $k \geq 1$ and a number $p > 1$, there is a constant $c = c(k,p) > 0$ so that for any $a,\ b \in \mathbf{R}^k$,

$$\big(|a|^{p-2}a - |b|^{p-2}b\big) \cdot (a-b) \geq c|a-b|^p, \quad \text{for } p \geq 2; \tag{2.7}$$

$$\big(|a|^{p-2}a - |b|^{p-2}b\big) \cdot (a-b) \geq c|a-b|^2(|a|+|b|)^{p-2}, \quad \text{for } 1 < p < 2. \tag{2.8}$$

We also need some properties of Hardy space and the space $\mathrm{BMO}(\mathbf{R}^n)$. By definition, $\mathrm{BMO}(\mathbf{R}^n)$ consists of all functions $f \in L^1_{\mathrm{loc}}(\mathbf{R}^n)$ with bounded mean oscillation (BMO):

$$\|f\|_{\mathrm{BMO}} = \sup\{\fint_{\mathbf{B}_r(x)} |f - \bar{f}_{x,r}|dy : x \in \mathbf{R}^n, r > 0\} < \infty, \tag{2.9}$$

where $\bar{f}_{x,r} = \fint_{\mathbf{B}_r(x)} f$ is mean value of f on $\mathbf{B}_r(x)$. Obviously, $L^\infty(\mathbf{R}^n) \subset \mathrm{BMO}(\mathbf{R}^n)$.

The Hardy space $\mathcal{H}^1(\mathbf{R}^n)$ is the set of all $g \in L^1(\mathbf{R}^n)$ such that

$$g^*(x) = \sup_{r>0} |\int_{\mathbf{R}^n} g(y)\phi_r(x-y)dy| \in L^1(\mathbf{R}^n),$$

with norm

$$\|g\|_{\mathcal{H}^1} = \|g^*\|_{L^1(\mathbf{R}^n)},$$

where ϕ is a fixed function in $C_0^\infty(\mathbf{B}_1)$ with $\int_{\mathbf{B}_1} \phi = 1$, and $\phi_r(x) = \frac{1}{r^n}\phi\left(\frac{x-y}{r}\right)$. Note that $\mathcal{H}^1(\mathbf{R}^n)$ is independent of the choice of ϕ; see [FS] or [Ss].

The delicate relation between $\mathcal{H}^1$ and BMO is contained in the following famous theorem of Fefferman-Stein [FS].

Theorem 2.5. $\mathcal{H}^1(\mathbf{R}^n)^* = \mathrm{BMO}(\mathbf{R}^n)$. *In particular, the integral $\int_{\mathbf{R}^n} fg$, which is well-defined for $f \in \mathcal{H}^1(\mathbf{R}^n) \cap C^\infty$ and $g \in \mathrm{BMO}(\mathbf{R}^n)$, can be extended to any $f \in \mathcal{H}^1(\mathbf{R}^n)$, and there is a constant $C = C(n)$ such that*

$$|\int_{\mathbf{R}^n} fg| \leq C\|f\|_{\mathcal{H}^1}\|g\|_{\mathrm{BMO}}.$$

In our application, functions are defined on $\Omega \subset \mathbf{R}^n$. We say a function $f \in L^1_{\rm loc}(\Omega)$ lies in $\mathcal{H}^1_{\rm loc}(\Omega)$ if each point in Ω has a neighborhood U ($\subset \bar{U} \subset \Omega$) on which f agrees with a function in $\mathcal{H}^1(\mathbf{R}^n)$. For such U, define

$$\|f\|_{\mathcal{H}^1(U)} = \inf\{\|g\|_{\mathcal{H}^1(\mathbf{R}^n)} : f|U = g|U\}. \tag{2.10}$$

Suppose $u \in W^{1,p}(\Omega, \mathbf{R}^m)$ is a solution of (2.1). We will assume that

$$f(\cdot, u, \nabla u) \in \mathcal{H}^1_{\rm loc}(\Omega). \tag{2.11}$$

More specifically, we assume that there is constant C such that for any ball $\mathbf{B}_\rho \subset \Omega$.

$$\|f(\cdot, u, \nabla u)\|_{\mathcal{H}^1\left(\mathbf{B}_{\frac{1}{2}\rho}\right)} \le C \int_{\mathbf{B}_\rho} |\nabla u|^p. \tag{2.12}$$

It follows that if $\psi \in W^{1,p}_0(\mathbf{B}_{\frac{1}{2}\rho}, \mathbf{R}^m) \cap \mathrm{BMO}$, then by Theorem 2.5,

$$|\int_{\mathbf{B}_\rho} f(\cdot, u, \nabla u) \cdot \psi| \le C\|\psi\|_{\rm BMO} \int_{\mathbf{B}_\rho} |\nabla u|^p. \tag{2.13}$$

Our main result in this section is

Theorem 2.6. *If $u \in W^{1,p}(\Omega, \mathbf{R}^m)$, $p \in (1, n]$, is a solution of (2.1) with the monotonicity property and $f(x, u, \nabla u)$ satisfies (2.11) and (2.12), then $u \in C^{1,\alpha}(\Omega \setminus Z, \mathbf{R}^m)$ for some $0 < \alpha < 1$ and closed subset $Z \subset \Omega$ with $\mathcal{H}^{n-p}(Z) = 0$. If $p = n$, then $u \in C^{1,\alpha}(\Omega, \mathbf{R}^m)$.*

The proof of this theorem is reduced to the following lemma.

Lemma 2.7. *There exist numbers $\varepsilon_0, \tau \in (0, 1)$, depending n, m, p, so that if u is as in Theorem 2.7 with $e(x, r) := r^{p-n} \int_{\mathbf{B}_r(x)} |\nabla u|^p \le \varepsilon_0$, then*

$$e(x, \tau r) \le \frac{1}{2} e(x, r).$$

Proof of Theorem 2.6. Take $\varepsilon = 2^{p-n}\varepsilon_0$, where ε_0 is as in Lemma 2.7. We claim that if $\mathbf{B}_{2\rho}(x) \subseteq \Omega$ and $e(x, 2\rho) \le \varepsilon$, then $u \in C^{1,\alpha}(\mathbf{B}_\rho(x), \mathbf{R}^m)$ for some $\alpha \in (0, 1)$.

Indeed, for any $y \in \mathbf{B}_\rho(x)$ and $r \in (0, \rho]$, the monotonicity of $e(x, r)$ in r implies

$$e(y, r) \le e(y, \rho) \le 2^{n-p} e(x, 2\rho) \le \varepsilon_0.$$

So Lemma 2.7 implies that for some $\tau \in (0, 1)$ and all $r \in (0, \rho]$, there holds

$$e(y, \tau r) \le \frac{1}{2}\, e(y, r). \tag{2.14}$$

For any $r \in (0, \rho]$, let $k \geq 1$ be an integer such that $r \in [\tau^k \rho, \tau^{k-1}\rho)$. Then it follows from (2.14) and the monotonicity of $e(y,r)$ that

$$e(y,r) \leq e(y, \tau^{k-1}\rho) \leq 2^{-k+1} e(y,\rho) \leq 2\varepsilon_0 \left(\frac{r}{\rho}\right)^{\theta},$$

where $\theta = \log_\tau \frac{1}{2}$. By Morrey's Lemma [MC, 3.5.2], u is $C^{\frac{\theta}{p}}$ on $\mathbf{B}_\rho(x)$. By Theorem 2.3, $u \in C^{1,\alpha}(\mathbf{B}_\rho(x), \mathbf{R}^m)$ for some $\alpha \in (0,1)$.

Let $Z = \{x \in \Omega : \liminf_{r\to 0} e(x,r) \geq \varepsilon\}$. Then $u \in C^{1,\alpha}(\Omega \setminus Z, \mathbf{R}^m)$, and that $\mathcal{H}^{n-p}(Z) = 0$ follows from a covering argument (see [G]). ∎

Proof of Lemma 2.7. We show that Lemma 2.7 holds for a $\tau \in (0, \frac{1}{8})$ satisfying $2^{2n+2} C \tau^p < 1$, where C is as in Theorem 2.4. For otherwise, there would be a sequence $\mathbf{B}_{r_i}(x_i) \subseteq \Omega$ such that

$$\lambda_i^p \equiv e(x_i, r_i) \downarrow 0, \quad \text{but} \quad e(x_i, \tau r_i) \geq \frac{1}{2}\lambda_i^p. \tag{2.15}$$

Define $v_i : \mathbf{B}_1 \to \mathbf{R}^m$ by

$$v_i(z) = \lambda_i^{-1}[u(x_i + r_i z) - \bar{u}_{x,r}], \quad z \in \mathbf{B}_1.$$

Then by Poincare inequality,

$$\int_{\mathbf{B}_1} |\nabla v_i|^p = 1, \quad \int_{B_1} |v_i|^p \leq C, \tag{2.16}$$

where $C = C(n,p)$. In terms of v_i, the second inequality of (2.15) becomes

$$\tau^{p-n} \int_{\mathbf{B}_\tau} |\nabla v_i|^p dz \geq \frac{1}{2}. \tag{2.17}$$

The boundedness of v_i in $W^{1,p}$ (2.16) implies that there is a subsequence $\{v_k\} \subseteq \{v_i\}$ such that

$$v_k \to v_0 \ \text{ in } L^p(\mathbf{B}_1, \mathbf{R}^m); \quad \nabla v_k \rightharpoonup \nabla v_0 \ \text{ weakly in } L^p(\mathbf{B}_1, \mathbf{R}^{nm}), \tag{2.18}$$

for some $v_0 \in W^{1,p}(\mathbf{B}_1, \mathbf{R}^m)$.

We want to show the strong convergence $v_k \to v_0$ in $W^{1,p}(\mathbf{B}_{\frac{1}{4}}, \mathbf{R}^m)$. For this, take a function $\xi \in C_0^1(\mathbf{B}_{\frac{1}{2}}, [0,1])$ so that $|\nabla \xi| \leq 2$ and $\xi = 1$ on $\mathbf{B}_{\frac{1}{4}}$.

First look at the case $p \geq 2$. Applying (2.7), we have

$$\begin{aligned} c\int_{\mathbf{B}_1} \xi |\nabla v_k - \nabla v_l|^p &\leq \int_{\mathbf{B}_1} [|\nabla v_k|^{p-2}\nabla v_k - |\nabla v_l|^{p-2}\nabla v_l] \cdot \nabla[(v_k - v_l)]\xi \\ &= \int_{\mathbf{B}_1} [|\nabla v_k|^{p-2}\nabla v_k - |\nabla v_l|^{p-2}\nabla v_l] \cdot [\nabla((v_k - v_l)\xi) + (v_k - v_l)\nabla\xi]. \end{aligned} \tag{2.19}$$

Using Hölder inequality together with (2.16), we have, as $k,\ l \to \infty$,

$$\left|\int_{\mathbf{B}_1} [|\nabla v_k|^{p-2}\nabla v_k - |\nabla v_l|^{p-2}\nabla v_l] \cdot [(v_k - v_l)\nabla\xi]\right| \leq C\|v_k - v_l\|_{L^p(\mathbf{B}_1)} \to 0. \tag{2.20}$$

Now we estimate the remaining terms in (2.19) by using (2.13) and following fact

$$\sup_{l,k} (\|v_l\xi\|_{\mathrm{BMO}} + \|v_k\xi\|_{\mathrm{BMO}}) < \infty. \tag{2.21}$$

(2.21) will be proved in a moment. Denote $\psi = (v_k - v_l)\xi$ and $\phi(\cdot) = \psi\left(\frac{\cdot - x_k}{r_k}\right)$. Assuming (2.21), then $\|\psi\|_{\mathrm{BMO}} = \|\phi\|_{\mathrm{BMO}}$ is bounded independent of k and l, and ϕ is supported in $\mathbf{B}_{\frac{1}{2}r_k}(x_k)$. By a change of variables, the equation (2.1) and (2.13), we get

$$\begin{aligned}
&|\int_{\mathbf{B}_1} |\nabla v_k|^{p-2}\nabla v_k \cdot [\nabla((v_k - v_l)\xi] = \frac{r_k^{p-n}}{\lambda_k^{p-1}} | \int_{\mathbf{B}_{r_k}(x_k)} |\nabla u|^{p-2}\nabla u \nabla\phi \; | \\
&= \frac{r_k^{p-n}}{\lambda_k^{p-1}} | \int_{\mathbf{B}_{r_k}(x_k)} f(\cdot, u, \nabla u)\cdot \phi \; | \le \frac{r_k^{p-n}}{\lambda_k^{p-1}} \|\phi\|_{\mathrm{BMO}} \int_{\mathbf{B}_{r_k}(x_k)} |\nabla u|^p \\
&= C\|\phi\|_{\mathrm{BMO}}\lambda_k \to 0, \quad \text{ as } k,\ l \to \infty.
\end{aligned}$$

Similarly, $|\int_{\mathbf{B}_1} |\nabla v_l|^{p-2}\nabla v_l \cdot [\nabla(v_k - v_l)\xi]| \to 0$ as $k,\ l \to \infty$. Those estmates, combined with (2.19)-(2.20), imply that $\int_{\mathbf{B}_{\frac{1}{4}}} |\nabla v_k - \nabla v_l|^p \to 0$ as $k,\ l \to \infty$. So ∇v_k is a Cauchy sequence and is strongly convergent in L^p.

Since v_k satisfies (2.16) and (2.17), its strong convergence implies that

$$\int_{\mathbf{B}_{\frac{1}{4}}} |\nabla v_0|^p dz \le 1, \quad \int_{\mathbf{B}_{\frac{1}{4}}} |v_0|^p dz \le C_0. \tag{2.22}$$

$$\int_{\mathbf{B}_{\frac{1}{4}}} |\nabla v_0|^{p-2}\nabla v_0 \cdot \nabla\varphi dz = 0$$

for all $\varphi \in C_0^1(\mathbf{B}_{\frac{1}{4}}, \mathbf{R}^m)$. So v_0 is a p-harmonic function on $\mathbf{B}_{\frac{1}{4}}$ and by Theorem 2.4 and (2.22),

$$\sup_{\mathbf{B}_{\frac{1}{8}}} |\nabla v_0|^p \le C \fint_{\mathbf{B}_{\frac{1}{4}}} |\nabla v_0|^p \le C\frac{4^n}{\omega_n}.$$

Thus for the chosen $0 < \tau < \frac{1}{8}$,

$$\tau^{p-n}\int_{\mathbf{B}_0} |\nabla v_0|^p dz \le C4^n\tau^p \le \frac{1}{4},$$

a contradiction to the limit of (2.17). This ends the case $p \ge 2$.

If $1 < p < 2$, we use (2.8) to get

$$\begin{aligned}
\int_{\mathbf{B}_{\frac{1}{4}}} |\nabla v_k - \nabla v_l|^p &\le C\left(\int_{\mathbf{B}_1} (|\nabla u| + |\nabla v|)^p\right)^{\frac{2}{2-p}} \\
&\left(\int_{\mathbf{B}_1} [|\nabla v_k|^{p-2}\nabla v_k - |\nabla v_l|^{p-2}\nabla v_l]\cdot\nabla[(v_k - v_l)]\xi\right)^{\frac{2}{p}}.
\end{aligned}$$

The rest of the proof is similar.

To complete the proof, we need to show (2.21). It suffices to show that $\sup_i \|v_i\xi\|_{\mathrm{BMO}} < \infty$. We follow the proof of Evans [EL] for the case $p=2$. Indeed, for $z \in \mathbf{B}_{\frac{7}{8}}$ and $r \in (0, \frac{1}{8}]$, by the monotonicity property of u, we have

$$
\begin{aligned}
r^{p-n}\int_{\mathbf{B}_r(z)} |\nabla v_i(y)|^p dy &= \lambda_i^{-p} r^{p-n}\int_{\mathbf{B}_r(z)} |\nabla u|^p (x_i + r_i y) r_i^p dy \\
&= \lambda_i^{-p}(r_i r)^{p-n}\int_{\mathbf{B}_{r_i r}(x_i+r_i z)} |\nabla u|^p(x) \\
&\le \lambda_i^{-p}(\frac{r_i}{8})^{p-n}\int_{\mathbf{B}_{\frac{r_i}{8}}(x_i+r_i z)} |\nabla u|^p(x) \\
&\le 8^{n-p}\lambda_i^{-p} r_i{}^{p-n}\int_{\mathbf{B}_{r_i}(x_i)} |\nabla u|^p(x) \le 8^{n-p}.
\end{aligned}
$$

From this and John-Nirenberg's inequality [JN], we get $\int_{\mathbf{B}_{\frac{7}{8}}(0)} |v_i|^n$ is bounded. Thus for $z \in \mathbf{B}_{\frac{6}{8}}$ and $r \in (0, \frac{1}{8}]$,

$$
r^{p-n}\int_{\mathbf{B}_r(z)} |v_i|^p \le C\left(\int_{\mathbf{B}_r(z)} |v_i|^n\right)^{\frac{p}{n}} \le C\left(\int_{\mathbf{B}_{\frac{7}{8}}(0)} |v_i|^n\right)^{\frac{p}{n}} \le C < \infty.
$$

Denote $w = v_i\xi$. For any $z \in \mathbf{B}_{\frac{3}{4}}(0)$ and $r \in (0, \frac{1}{8}]$, we obtain from Poincare's inequality and the above two estimates,

$$
\begin{aligned}
\tag{2.23}
&⨍_{\mathbf{B}_r(z)} |w - \bar{w}_{z,r}| \le C r^{1-n}\int_{\mathbf{B}_r(z)} |\nabla w| \\
&\le C\left(r^{p-n}\int_{\mathbf{B}_r(z)} |\nabla w|^p\right)^{\frac{1}{p}} \\
&\le C\left(r^{p-n}\int_{\mathbf{B}_r(z)} |\nabla v_i|^p + |v_i|^p\right)^{\frac{1}{p}} \le C < \infty.
\end{aligned}
$$

This estimate also holds for $z \in \mathbf{B}^n \setminus \mathbf{R}_{\frac{3}{4}}(0)$ or $r > \frac{1}{8}$. By definition (2.9), we see $\sup_i \|v_i\xi\|_{\mathrm{BMO}} < \infty$. Thus (2.21) is proved. ■

§ 3. Applications

To apply Theorem 2.6, we need to know when the right hand side f is in $\mathcal{H}^1$ and satisfies (2.13). Coiffman, Lions, Meyer and Semmes give the following criterion [CLMS1,2]:

Suppose that $B \in L^p(\mathbf{R}^n, \mathbf{R}^n)$ *and* $E \in L^{p'}(\mathbf{R}^n, \mathbf{R}^n)$, *where* $1 < p < \infty$ *and* $p' = \frac{p}{p-1}$, *so that in distributional sense* $\mathrm{curl}(B) = (\frac{\partial B_i}{\partial x_j} - \frac{\partial B_j}{\partial x_i})_{ij} = 0$ *(curl free) and* $\mathrm{div}(E) = 0$ *(divergence free). Then* $E \cdot B \in \mathcal{H}^1(\mathbf{R}^n)$ *and for some constant* $C(n,p)$

$$
\|E \cdot B\|_{\mathcal{H}^1} \le C\|E\|_{L^{p'}}\|B\|_{L^p}.
$$

In particular, it applies to the case when $E = \nabla u$, where $u \in W^{1,p}(\mathbf{R}^n)$.

A local version of the above criterion also holds. It directly follows from the proof in [CLMS2] (pages 9-10). In particular, if $u \in W_0^{1,p}(\mathbf{B}_1)$ and $E \in L^{p'}(\mathbf{B}_1, \mathbf{R}^n)$ is divergence free, then we also have $\nabla u \cdot E \in \mathcal{H}^1$ (E is extended so that $E = 0$ on $\mathbf{R}^n \setminus \mathbf{B}_1$). Thus by the definition (2.10), $\nabla u \cdot E \in \mathcal{H}^1_{\text{loc}}(\mathbf{B}_1)$ and

$$\|\nabla u \cdot E\|_{\mathcal{H}^1(\mathbf{B}_{\frac{1}{2}})} \le \|\nabla u \cdot E\|_{\mathcal{H}^1} \le C\|E\|_{L^{p'}(\mathbf{B}_1)}\|\nabla u\|_{L^p(\mathbf{B}_1)}, \tag{3.0}$$

for a constant C independent of u and E.

If u does not have compact support, we may consider $g \equiv (u - \bar{u})\xi$, where $\bar{u} = ⨍_{\mathbf{B}_1} u$ and $\xi \in C_0^1(\mathbf{B}_{\frac{3}{4}}, [0,1])$ is a cut-off function with $\xi = 1$ on $\mathbf{B}_{\frac{1}{2}}$ and $|\nabla \xi| \le 6$. Note that $\nabla g \cdot E = \nabla u \cdot E$ on $\mathbf{B}_{\frac{1}{2}}$ and by Poincare's inequality $\|\nabla g\|_{L^p(\mathbf{B}_1)} \le C\|\nabla u\|_{L^p(\mathbf{B}_1)}$. Then by definition (2.10), and (2.13) applied to $\nabla g \cdot E$, we have that $\nabla u \cdot E$ also satisfies (3.0). By rescaling, we obtain

Proposition 3.1. *Suppose that $u \in W^{1,p}(\mathbf{B}_\rho)$, $E \in L^{p'}(\mathbf{B}_\rho, \mathbf{R}^n)$ is divergence free, then*

$$\|\nabla u \cdot E\|_{\mathcal{H}^1(\mathbf{B}_{\frac{1}{2}\rho})} \le C\|E\|_{L^{p'}(\mathbf{B}_\rho)}\|\nabla u\|_{L^p(\mathbf{B}_\rho)}.$$

Theorem 3.2. *Suppose $p \in (1,n)$ and $\mathbf{S} = \mathbf{S}^{m-1}$. If $u \in W^{1,p}(\Omega, \mathbf{S})$ is a p-harmonic map with the monotonicity property, then $u \in C^{1,\alpha}(\Omega \setminus Z, \mathbf{S})$ for some $0 < \alpha < 1$ and closed $Z \subset \Omega$ with $\mathcal{H}^{n-p}(Z) = 0$. If $p = n$, then $u \in C^{1,\alpha}(\Omega, \mathbf{S})$.*

For the proof, we first give another description on p-harmonic maps to spheres.

Proposition 3.3. *If $u \in W^{1,p}(\Omega, \mathbf{R}^m)$, $|u| = 1$, then u is a p-harmonic map to $\mathbf{S}$ if and only if for all $1 \le i, j \le n$, the vector function*

$$E^{ij} = (E^{ij}_\alpha)_{1 \le \alpha \le m} = \left(|\nabla u|^{p-2} u^i_\alpha u^j - |\nabla u|^{p-2} u^j_\alpha u^i\right)_{1 \le \alpha \le m} \tag{3.2}$$

is divergence free, that is, $\int_\Omega E^{ij}_\alpha \varphi_\alpha = 0$ for every $\varphi \in C^1_0(\Omega)$. (As usual, repeated indices are summed.)

Proof : Suppose that u is p-harmonic, then by (1.3), for every $\varphi \in C^1_0(\Omega)$,

$$\begin{aligned} \int_\Omega E^{ij}_\alpha \varphi_\alpha &= \int_\Omega |\nabla u|^{p-2} u^i_\alpha u^j \varphi_\alpha - |\nabla u|^{p-2} u^j_\alpha u^i \varphi_\alpha \\ &= \int_\Omega |\nabla u|^{p-2} u^i_\alpha (u^j \varphi)_\alpha - |\nabla u|^{p-2} u^j_\alpha (u^i \varphi)_\alpha \\ &= \int_\Omega |\nabla u|^p u^i u^j \varphi - |\nabla u|^p u^j u^i \varphi = 0. \end{aligned} \tag{3.3}$$

So E^{ij} is divergence free. Conversely, suppose (3.3) holds for all $\varphi \in C^1_0(\Omega)$. By approximation, it holds with φ replaced by $u^i\varphi \in W^{1,p}_0(\Omega, \mathbf{R}^m)$. It follows from $\sum_i u^i u^i = 1$ that $\sum_i u^i_\alpha u^i = 0$ and

$$\begin{aligned} 0 = \int_\Omega E^{ij}_\alpha (u^i\varphi)_\alpha &= \int_\Omega |\nabla u|^{p-2} u^i_\alpha u^j (u^i\varphi)_\alpha - |\nabla u|^{p-2} u^j_\alpha u^i (u^i \varphi)_\alpha \\ &= \int_\Omega |\nabla u|^p u^j \varphi - |\nabla u|^{p-2} u^j_\alpha \varphi_\alpha. \end{aligned}$$

This is exactly (1.3). So u is p-harmonic. ∎

Proof of Theorem 3.2. We show that $f = u|\nabla u|^p$ satisfies (2.11) and (2.12).

For $k \in \{1, \dots, n\}$, we write the k-th component of f as

$$f^k = u^k|\nabla u|^p = u^i_\alpha \left(u^i_\alpha u^k - u^k_\alpha u^i\right)|\nabla u|^{p-2} \equiv \nabla u^i \cdot E^{ik}$$

where we have used the fact that $\sum_i u^i_\alpha u^i = 0$. By Proposition 3.3, E^{ik} are of divergence free and $|E^{ik}| \le |\nabla u|^{p-1}$. By Proposition 3.1, we obtain that

$$\|f\|_{\mathcal{H}^1(\mathbf{B}_{\frac{1}{2}\rho})} \le C\|E^{ik}\|_{L^{p'}(\mathbf{B}_\rho)}\|\nabla u\|_{L^p(\mathbf{B}_\rho)} \le C\|\nabla u\|^p_{L^p(\mathbf{B}_\rho)}.$$

■

Now we turn to n-harmonic maps with constant volume. For $u = (u^1, \dots, u^{n+1}) \in C^1(\Omega, R^{n+1})$, define

$$V(u) = \int_\Omega u^1 J(u^2, \dots, u^{n+1}), \tag{3.4}$$

where $J(u^2, \dots, u^{n+1}) = \dfrac{\partial(u^2, \dots, u^{n+1})}{\partial(x^1, \dots, x^n)}$.

Proposition 3.4. *V is well-defined for every $u = (u^1, \dots, u^{n+1}) \in W^{1,n}_0(\Omega) \times W^{1,n}(\Omega, \mathbf{R}^n)$ and for some $C = C(n, \Omega)$, there holds*

$$|V(u)| \le C\ \Pi_{i=1}^{n+1}\|\nabla u^i\|_{L^n(\Omega)}. \tag{3.5}$$

Proof. We first prove the estimate (3.5) for $u \in C^1$, for which $V(u)$ is obviously well-defined and finite. Extend u^1 as $\tilde{u}^1$ on $\mathbf{R}^n$ so that $\tilde{u}^1 = 0$ on $\mathbf{R}^n \setminus \Omega$. It is easy to see that (cf. (2.23)) $\tilde{u}^1 \in \mathrm{BMO}(\mathbf{R}^n)$ and for some $C(n)$,

$$\|\tilde{u}^1\|_{\mathrm{BMO}} \le C\ \|\nabla\tilde{u}^1\|_{L^n(\Omega)} = C\|\nabla u^1\|_{L^n(\Omega)}.$$

For $i \ge 2$, denote $\bar{u}^i = \fint_\Omega u^i$. By extension theorem [AR], there is a function $w^i \in W^{1,n}(\mathbf{R}^n)$ such that $w^i = u^i - \bar{u}^i$ and $\|\nabla w^i\|_{L^n(\mathbf{R}^n)} \le C_1\|u^i - \bar{u}^i\|_{W^{1,n}(\Omega)} \le C_2\|\nabla u^i\|_{L^n(\Omega)}$, where C_1 and C_2 depend only on n and Ω. Define $\tilde{u}^i = w^i + \bar{u}^i$. Then $\tilde{u}^i$ extends u^i and satisfies

$$\|\nabla\tilde{u}^i\|_{L^n(\mathbf{R}^n)} \le C\|\nabla u^i\|_{L^n(\Omega)}. \tag{3.6}$$

By Theorem II.1 (1), $J(\tilde{u}^2, \dots, \tilde{u}^{n+1}) \in \mathcal{H}^1(\mathbf{R}^n)$ and for some $C = C(n, \Omega)$,

$$\|J(\tilde{u}^2, \dots, \tilde{u}^{n+1})\|_{\mathcal{H}^1} \le C\ \Pi_{i=2}^{n+1}\|\nabla u^i\|_{L^n(\Omega)}.$$

Now apply Theorem 2.5 to get

$$\begin{aligned} |\int_\Omega u^1 J(u^2, \dots, u^{n+1})| &= |\int_{\mathbf{R}^n} \tilde{u}^1 J(\tilde{u}^2, \dots, \tilde{u}^{n+1})| \\ &\le C\ \|\tilde{u}^1\|_{\mathrm{BMO}}\|J(\tilde{u}^2, \dots, \tilde{u}^{n+1})\|_{\mathcal{H}^1} \\ &\le C\ \Pi_{i=1}^{n+1}\|\nabla u^i\|_{L^n(\Omega)}. \end{aligned}$$

This estimate shows that V is continuous in the norm of $W^{1,n}(\Omega, \mathbf{R}^{n+1})$; so V extends to $W^{1,n}_0(\Omega) \times W^{1,n}(\Omega, \mathbf{R}^n)$, the closure of $C^1_0(\Omega, \mathbf{R}) \times C^1(\Omega, \mathbf{R}^n)$ in $W^{1,n}(\Omega, \mathbf{R}^{n+1})$, and the inequality also holds. ■

Corollary 3.5.

(a). For $u \in W^{1,n}(\Omega, \mathbf{R}^{n+1})$ and $v \in W_0^{1,n}(\Omega, \mathbf{R}^{n+1})$, denote $\mathbf{J}(u) = \partial_1 u \wedge \ldots \wedge \partial_n u$. Then $\int_\Omega v \cdot \mathbf{J}(u)$ is well defined and

$$|\int_\Omega v \cdot \mathbf{J}(u)| \leq C \|\nabla v\|_{L^n(\Omega)} \|\nabla u\|_{L^n(\Omega)}^n, \tag{3.7}$$

for some constant $C(n, \Omega)$.

(b). $\mathcal{V}(u) \equiv \int_\Omega u \cdot \mathbf{J}(u)$ is well-defined for every $u \in W^{1,n}(\Omega, \mathbf{R}^{n+1})$ with $u|\partial\Omega$ in L^∞.

Proof. Note that

$$\int_\Omega v \cdot \mathbf{J}(u) = \sum_{i=1}^{n+1} (-1)^i v^i J(u^1, \cdots, u^{i-1}, u^{i+1}, \cdots, u^{n+1}).$$

So (3.7) follows from (3.5).

For u as in (b), let $\bar{u}$ be the harmonic extension of $u|\partial\Omega$ on Ω. Then $\bar{u} \in C^1(\Omega)$ and by maximum principle, $\|\bar{u}\| \leq \max_{\partial\Omega} \|u\|$. Note $\int_\Omega \bar{u} \cdot \mathbf{J}(u)$ is well-defined since $\bar{u}$ is bounded and $\mathbf{J}(u) \in L^1$, while $\int_\Omega (u - \bar{u}) \cdot \mathbf{J}(u)$ is well-defined since $u - \bar{u} \in W_0^{1,n}(\Omega, R^{n+1})$ and by part (a). So $\mathcal{V}(u) = \int_\Omega u \cdot \mathbf{J}(u) = \int_\Omega \bar{u} \cdot \mathbf{J}(u) + \int_\Omega (u - \bar{u}) \cdot \mathbf{J}(u)$ is also well-defined. ■

Thus for a given bounded boundary data $\phi|\partial\Omega$, the image of any function $u \in W^{1,n}(\Omega, \mathbf{R}^m)$ with $u|\partial\Omega = \phi$ spans a cone with finite volume $\mathcal{V}(u)$. The critical points of $\int_\Omega |\nabla u|^n$ subject to prescribed volume are called *n-harmonic maps with constant volume*, and satisfy (1.5). Here we apply Theorem 2.6 to obtain the regularity of such n-harmonic maps.

Theorem 3.6. *Suppose that $u \in W^{1,n}(\Omega, \mathbf{R}^{n+1})$ is a weak solution of the equation (1.5). Then $u \in C^{1,\alpha}(\Omega, \mathbf{R}^{n+1})$ for some $\alpha \in (0, 1)$.*

Proof. Again we need to verify that $f = H\mathbf{J}(u)$ satisfies (2.11) and (2.12).

For any $\mathbf{B}_r \subset \Omega$, there is an extension $\tilde{u} \in W^{1,n}(\mathbf{R}^n, \mathbf{R}^{n+1})$ of $u|\mathbf{B}_r$ such that $\|\nabla \tilde{u}\|_{L^n(\mathbf{R}^n)} \leq C\|\nabla u\|_{L^n(\mathbf{B}_r)}$, as in (3.6). In this case, C depends only on n. Let $\lambda = ⨍_{\mathbf{B}_{2r}} u$ and consider $v = \xi(\tilde{u} - \lambda)$, where $\xi \in C_0^1(\mathbf{B}_{2r}, [0, 1])$ is a cut-off function with $\xi = 1$ on $\mathbf{B}_r$ and $|\nabla \xi| \leq \frac{2}{r}$. Then it is not hard to see that $H\mathbf{J}(u) = H\mathbf{J}(v)$ on $\mathbf{B}_r$ and $H\mathbf{J}(v)$ is a $\mathcal{H}^1$ function with

$$\|\mathbf{J}(v)\|_{\mathcal{H}^1(\mathbf{B}_r)} \leq C\|\nabla v\|_{L^n(\mathbf{B}_{2r})}^n \leq C\|\nabla u\|_{L^n(\mathbf{B}_r)}^n.$$

So (2.12) holds. ■

§ 4. Boundary regularity

Theorem 4.1. *Suppose that $\partial\Omega$ is Lipschitz, $\phi \in C^0(\partial\Omega, \mathbf{R}^m)$. If $u \in C^1(\Omega, R^m)$ is a weak solution of*

$$\mathrm{div}(|\nabla u|^{n-2} \nabla u) = f(x, u, \nabla u), \tag{4.1}$$

where f and u satisfy (2.2) and (2.3), and $u = \phi$ on $\partial\Omega$ (in trace sense), then $u \in C^0(\bar{\Omega}, \mathbf{R}^m) \cap C^{1,\alpha}(\Omega, \mathbf{R}^m)$ for some $\alpha \in (0, 1)$.

We first recall a property for functions in $W^{1,n}$. For $x \in \mathbf{B}_\rho(x_0)$, we may write x in polar coordinates as $x = r\theta$, where $r = |x - x_0|$, $\theta \in \mathbf{S} := \mathbf{S}^{m-1}$.

Lemma 4.2. *If $u \in W^{1,n}(\mathbf{B}_{2\rho}, \mathbf{R}^m)$, then there exist $r \in (\rho, 2\rho)$ and $C = C(n, m)$ such that*

$$\int_{\mathbf{S}} |\nabla_\theta u|^n(r\theta) d\mathbf{S} \leq 2 \int_{\mathbf{B}_{2\rho}} |\nabla u|^n dx, \tag{4.2}$$

$$|u(r\theta_1) - u(r\theta_2)| \leq C|\theta_1 - \theta_2|^{\frac{1}{n}} \|\nabla u\|_{L^n(\mathbf{B}_{2\rho})}. \tag{4.3}$$

for all $\theta_1, \theta_2 \in \mathbf{S}$.

Proof. Note that $|\nabla u|^2 = |u_r|^2 + r^{-2}|u_\theta|^2$, where $u_\theta = \nabla_\theta u$. We have

$$\int_{\mathbf{B}_{2\rho}} |\nabla u|^n \geq \int_\rho^{2\rho} \int_{\mathbf{S}} [|u_r|^2 + r^{-2}|u_\theta|^2]^{\frac{n}{2}} r^{n-1} dr d\mathbf{S} \geq \int_\rho^{2\rho} r^{-1} \int_{\mathbf{S}} |u_\theta|^n dr d\mathbf{S}$$

By Fubini's Theorem, there is a $r \in (\rho, 2\rho)$ such that

$$\int_{\mathbf{B}_{2\rho}} |\nabla u|^n \geq \frac{\rho}{r} \int_{\mathbf{S}} |u_\theta|^n(r\theta) d\mathbf{S} \geq \frac{1}{2} \int_{\mathbf{S}} |u_\theta|^n(r\theta) d\mathbf{S},$$

this shows (4.2). By (4.2) and Sobolev's embedding theorem $W^{1,n}(\mathbf{S}, \mathbf{R}^m) \subset C^{0,\alpha}(\mathbf{S}, \mathbf{R}^m)$, (4.3) follows, with $\alpha = \frac{1}{n}$. ■

Theorem 4.3. *There are constants $\varepsilon > 0$ and C depending only on n, m, M but not on Ω so that if u is as in Theorem 4.1, $\sup_\Omega |u| \leq M$ and $\int_\Omega |\nabla u|^n < \varepsilon_0$, then*

$$\sup_\Omega |u - p| \leq C \sup_{\partial\Omega} |u - P| + C\|\nabla u\|_{L^n(\Omega)}.$$

Proof. Let ε be as in Theorem 2.3 (with $p = n$), then there is a constant $C = C(n, m, \mu(M))$ such that for every ball $\mathbf{B}_r \subset \Omega$ we have

$$\sup_{\mathbf{B}_{\frac{r}{2}}} |\nabla u| \leq Cr^{-1}\|\nabla u\|_{L^n(\Omega)}. \tag{4.4}$$

Let $L = \sup_\Omega |u - P|$ and $p \in \Omega$ be a point such that

$$|u(p) - P| \geq \frac{3}{4}L. \tag{4.5}$$

Denote $\rho = \mathrm{dist}(p, \partial\Omega) > 0$. If $\|\nabla u\|_{L^n(\Omega)} \geq L$, then the conclusion already holds. Otherwise we proceed as follows.

From (4.5), it follows

$$\sup_{\mathbf{B}_{\frac{\rho}{2}}(p)} |\nabla u| \le \frac{C}{\rho} L. \tag{4.6}$$

Let $r_1 = \frac{\rho}{2C}$. Then from (4.4)-(4.6) we have for $q \in \bar{\mathbf{B}}_{r_1}(p)$, there holds

$$|u(q) - P| > \frac{L}{4}. \tag{4.7}$$

Since Ω is Lipschitz, there is, by Theorem 4.32 in [AR], an extension $\tilde{u}$ of u to $\mathbf{R}^m$ such that $\|\nabla \tilde{u}\|_{L^n(\mathbf{R}^n)} \le C\|\nabla u\|_{L^n(\Omega)}$ for some $C = C(n, \Omega)$. By Lemma 4.2, applied to $\tilde{u}$ on $\mathbf{B}_{2\rho}(p)$, there is a $r_2 \in (\rho, 2\rho)$ such that

$$|u(r_2\theta_1) - u(r_2\theta_2)| \le C\|\nabla u\|_{L^n(\Omega)} |r_2\theta_1 - r_2\theta_2|^{\frac{1}{n}} \le C_1 \|\nabla u\|_{L^n(\Omega)}, \tag{4.8}$$

for all $r_2\theta_1, r_2\theta_2 \in \mathbf{S}_{r_2}(p)$. Take a $\theta_1 \in \mathbf{S}$ such that $|u(r_1\theta_1) - u(r_2\theta_1)| = \inf_{\theta\in\mathbf{S}} |u(r_1\theta) - u(r_2\theta)|$. By using Hölder's inequality and the fact $r_2 \le 4Cr_1$, we have

$$\begin{aligned} |u(r_1\theta_1) - u(r_2\theta_1)| &\le \frac{1}{\omega_n} \int_{\mathbf{S}} \int_{r_1}^{r_2} |u_r(r\theta)| dr d\mathbf{S} \\ &\le \frac{1}{\omega_n r_1^{n-1}} \int_{\mathbf{B}_{r_2}(p)} |u_r| \le \frac{1}{\omega_n r_1^{n-1}} \|\nabla u\|_{L^n(\mathbf{B}_{r_2}(p))} [\omega_n r_2]^{n-1} \le C\|\nabla u\|_{L^n(\Omega)}. \end{aligned} \tag{4.9}$$

Now take an $r_2\theta_2 \in \mathbf{S}_{r_2}(p) \cap \partial\Omega \ne \emptyset$. Then from (4.7)-(4.9) we have

$$\begin{aligned} \frac{L}{4} &\le |u(r_1\theta_1) - P| \\ &\le |u(r_1\theta_1) - |u(r_2\theta_1)| + |u(r_2\theta_1) - u(r_2\theta_2)| + |u(r_2\theta_2) - P| \\ &\le C\|\nabla u\|_{L^n(\Omega)} + \sup_{\partial\Omega} |u - P|. \end{aligned}$$

By the definition of L, we get the desired inequality. ■

Proof of Theorem 4.1. Extending u to $\mathbf{R}^n$, we may assume the u is defined on $\mathbf{R}^n$ with $\int_{\mathbf{R}^n} |\nabla u|^n < \infty$. Let $p \in \partial\Omega$ and denote $P = u(p)$. For any given small number $\delta > 0$, we may choose $\rho > 0$ such that

$$\int_{\mathbf{B}_\rho(p)} |\nabla u|^n \le \delta, \quad |u(q) - P| \le \delta, \tag{4.10}$$

for all $q \in \mathbf{B}_\rho(p) \cap \partial\Omega$. By Lemma 4.2 and (4.10), there is an $r \in (\frac{\rho}{2}, \rho)$ such that

$$\int_{\mathbf{S}_r(p)} |\nabla u|^n \le 2\delta, \quad |u(q_1) - u(q_2)| \le C\delta^{\frac{1}{n}}, \tag{4.11}$$

for all $q_1, q_2 \in \mathbf{S}_r(p)$. Let $\Sigma = \mathbf{B}_r(p) \cap \Omega$ and choose $\delta \le \varepsilon$ for the ε in Theorem 4.3. Then Σ is Lipschitz and by (4.10)-(4.11)

$$\sup_{\partial\Sigma} |u - P| \le \delta + C\delta^{\frac{1}{n}}. \tag{4.12}$$

By Theorem 4.3 (with $\Omega = \Sigma$) and (4.12), we have

$$\begin{aligned} \sup_{\Sigma} |u - P| &\le C \sup_{\partial\Sigma} |u - P| + C\|\nabla u\|_{L^n(\Sigma)} \\ &\le C\delta + C\delta^{\frac{1}{n}}. \end{aligned}$$

So u is continuous at p. ■

§ 5. Isolated singularities are removable

Theorem 5.1. *If $u \in C^1(\mathbf{B}_1 \setminus \{0\}, \mathbf{R}^m)$ is an solution of*

$$\operatorname{div}(|\nabla u|^{n-2}\nabla u) = f(x, u, \nabla u), \tag{5.1}$$

finite energy $\int_\Omega |\nabla u|^n < \infty$, where f and u satisfy (2.2) and (2.3), then u extends to a solution in $C^{1,\alpha}(\mathbf{B}_1, \mathbf{R}^m)$ for some $\alpha \in (0,1)$.

Corollary 5.2.

(a). *If $u \in C^1(\mathbf{B}_1 \setminus \{0\}, N)$ is an n-harmonic map with finite energy, then u extends to an n-harmonic map in $C^{1,\alpha}(\mathbf{B}_1, N)$ for some $\alpha \in (0,1)$.*

(b). *If $u \in C^1(\mathbf{B}_1 \setminus \{0\}, \mathbf{R}^{n+1})$ satisfies the equation*

$$\operatorname{div}(|\nabla u|^{n-2}\nabla u) = g\mathbf{J}(u)$$

on $\mathbf{B}_1 \setminus \{0\}$, where $g \in C^1(\mathbf{B}_1)$, then $u \in C^{1,\alpha}(\mathbf{B}_1, \mathbf{R}^{n+1})$ for some $\alpha \in (0,1)$.

Theorem 5.3. *If $u \in C^1(\mathbf{B}_1 \setminus \{0\}, \mathbf{R}^m)$ is in Theorem 5.1, then*

$$\int_{\mathbf{S}_r} |\nabla u|^n \le n^{\frac{n}{2}} \int_{\mathbf{S}_r} |u_r|^n, \quad \int_{\mathbf{S}_r} |\nabla u|^n \le \left(\frac{n}{n-1}\right)^{\frac{n}{2}} \int_{\mathbf{S}_r} |u_\theta|^n. \tag{5.2}$$

where $u_r = \frac{\partial u}{\partial r}$, and $u_\theta = \nabla_{\mathbf{S}} u$.

Proof : Using Hölder's inequality in (2.6) (cf. Remark there), we get

$$\int_{\mathbf{S}_r} |\nabla u|^n \le n \left(\int_{\mathbf{S}_r} |\nabla u|^n\right)^{\frac{n-2}{n}} \left(\int_{\mathbf{S}_r} |u_r|^n\right)^{\frac{2}{n}}.$$

This gives the first part of (5.2). Replacing $|\nabla u|^n$ by $|\nabla u|^{n-2}[|u_\theta|^2 + |u_r|^2]$ in (2.6), we get

$$\int_{\mathbf{S}_r} |\nabla u|^{n-2}|u_\theta|^2 = (n-1)\int_{\mathbf{S}_r} |\nabla u|^{n-2}|u_r|^2 = \frac{n-1}{n}\int_{\mathbf{S}_r} |\nabla u|^n.$$

Using Hölder inequality again to get the second part of (5.2). ■

Proof of Theorem 5.1. Since u has finite energy, we may take $s > 0$ such that that $\int_{\mathbf{B}_{2s}} |\nabla u|^n \le \varepsilon$, where ε is as in Theorem 2.3. Thus for any $x \in \mathbf{B}_s \setminus \{0\}$, we may apply Theorem 2.3 to $\mathbf{B}_r(x) \subset \mathbf{B}_{2s}$ with $r = \frac{|x|}{2}$ to get for some constant C,

$$|\nabla u|(x) \le \frac{C}{|x|}\left(\int_{\mathbf{B}_{2|x|}} |\nabla u|^n\right)^{\frac{1}{n}}. \tag{5.3}$$

Fix a $t \in (0, s]$ and define a function $q : (0, t] \to \mathbf{R}^m$ such that

$$q(r) = a_i \log r + b_i, \quad \text{for } r \in [\frac{t}{2^{i+1}}, \frac{t}{2^i}], \quad i = 0, 1, 2, \dots,$$

where a_i and b_i are chosen so that q is continuous and $q(\frac{t}{2^i}) = \fint_{\mathbf{S}_1} u(\frac{t}{2^i}\cdot)$.

Note that $q(|x|)$ is an n-harmonic function on $\{\frac{t}{2^{i+1}} \leq |x| \leq \frac{t}{2^i}\}$, that is,

$$\operatorname{div}(|\nabla q|^{n-2}\nabla q) = 0. \tag{5.4}$$

Note that $q(r)$ is componentwise monotone, then for any $\varphi \in \mathbf{S}_1$ and $\frac{t}{2^{i+1}} \leq r \leq \frac{t}{2^i}$, we use (5.3) and the definition of q to estimate

$$\begin{aligned} |q(r) - u(r\varphi)| &= |q(\frac{t}{2^{i+1}}) - q(\frac{t}{2^i})| + |q(\frac{t}{2^i}) - u(r\varphi)| \\ &\leq \fint_{\mathbf{S}_1} |u(\frac{t}{2^i}\cdot) - u(\frac{t}{2^{i+1}}\cdot)| + \fint_{\mathbf{S}_1} |u(\frac{t}{2^i}\cdot) - u(r\varphi)| \\ &\leq 2^{-i+2} \max_{\frac{t}{2^{i+1}} \leq r \leq \frac{t}{2^i}} |\nabla u(r\cdot)| \\ &= 2^{-i+2}2^{i+1}C\left(\int_{\mathbf{B}_{2s}} |\nabla u|^n\right)^{\frac{1}{n}} \leq 8C\varepsilon^{\frac{1}{n}}. \end{aligned} \tag{5.5}$$

By (2.7) and integration by parts, we get

$$\begin{aligned} c\int_{\mathbf{B}_1} |\nabla q - \nabla u|^n &\leq \int_{\mathbf{B}_t} \left(|\nabla q|^{n-2}\nabla q - |\nabla u|^{n-2}\nabla u\right)(\nabla q - \nabla u) \\ &\leq -\int_{\mathbf{B}_t} \operatorname{div}\left(|\nabla q|^{n-2}\nabla q - |\nabla u|^{n-2}\nabla u\right)(q-u) + \\ &+ \sum_{i=0}^{\infty}\int_{\mathbf{S}_r} \left(|\nabla q|^{n-2}q_r - |\nabla u|^{n-2}u_r\right)(q-u)\Big|_{r=\frac{t}{2^{i+1}}}^{r=\frac{t}{2^i}}. \end{aligned} \tag{5.6}$$

The first integral in (5.6) is simplified by (5.1) and (5.4). Those terms containing q_r in the second sum are zero since $q(\frac{t}{2^i}) = \fint_{\mathbf{S}_t} u(\frac{t}{2^i}\cdot)$. Since u_r is continuous, the other terms that contain the factor u_r are canceled successively, with only the first one remained. So we have

$$c\int_{\mathbf{B}_t} |\nabla q - \nabla u|^n \leq -\int_{\mathbf{B}_t} f(x,u,\nabla u)(q-u) + \int_{\mathbf{S}_t} |\nabla u|^{n-2}u_r(q-u). \tag{5.7}$$

Using (2.2) and (5.5), we get

$$|\int_{\mathbf{B}_t} f(x,u,\nabla u)(q-u)| \leq C\sup|q-u|\int_{\mathbf{B}_t} |\nabla u|^n \leq C\varepsilon^{\frac{1}{n}}\int_{\mathbf{B}_t} |\nabla u|^n. \tag{5.8}$$

From the Poincare inequality and $q(t) = \fint_{\mathbf{S}} u(t\cdot)$, we obtain $\int_{\mathbf{S}_t} |q-u|^n \leq C(n)t^n \int_{\mathbf{S}_t} |u_\theta|^n$. Thus by Hölder's inequality and using $|u_r|, |u_\theta| \leq |\nabla u|$, we have

$$|\int_{\mathbf{S}_t} |\nabla u|^{n-2}u_r(q-u)| \leq C(n)t\left(\int_{\mathbf{S}_t} |\nabla u|^n\right)^{\frac{n-1}{n}}\left(\int_{\mathbf{S}_t} |u_\theta|^n\right)^{\frac{1}{n}} \leq Ct\int_{\mathbf{S}_t} |\nabla u|^n. \tag{5.9}$$

On the other hand, the fact that q is independent of θ and (5.2) imply

$$\begin{aligned} \int_{\mathbf{B}_t} |\nabla q - \nabla u|^n &= \int_{\mathbf{B}_t} \left(|u_\theta|^2 + |q_r - u_r|^2\right)^{\frac{n}{2}} \\ &\geq \int_{\mathbf{B}_t} |u_\theta|^n \geq \left(\frac{n-1}{n}\right)^{\frac{n}{2}}\int_{\mathbf{B}_t} |\nabla u|^n. \end{aligned} \tag{5.10}$$

From (5.6)-(5.10), we see that if we start with a small ε, then there is a constant $1 > \tau > 0$ such that

$$\tau \int_{\mathbf{B}_t} |\nabla u|^n \leq t \int_{\mathbf{S}_t} |\nabla u|^n,$$

which implies that for $0 < t < s$,

$$\int_{\mathbf{B}_t} |\nabla u|^n \leq t^\tau \int_{\mathbf{B}_s} |\nabla u|^n \leq \varepsilon t^\tau. \tag{5.11}$$

For $x \in \mathbf{B}_{\frac{s}{2}}$, going back to (5.3) and applying (5.11) with $t = 2|x|$, we get

$$|\nabla u(x)| \leq \frac{C}{|x|} \left(\int_{\mathbf{B}_{2|x|}} |\nabla u|^n \right)^{\frac{1}{n}} \leq C_1 |x|^{\frac{\tau}{n} - 1}.$$

This implies that for $p = \frac{n}{1 - \frac{\tau}{n}} > n$, $u \in W^{1,p}(\mathbf{B}_1, \mathbf{R}^m) \subset C^{0, \frac{\tau}{n}}(\mathbf{B}_1, \mathbf{R}^m)$. Apply Theorem 2.3 (a) to finish the proof. ■

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