Generalized Riccati Equations Arising in Stochastic Games

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Abstract. We study a class of rational matrix differential equations that generalize the Riccati differential equations. The generalization involves replacing positive definite "weighting" matrices in the usual Riccati equations with either semidefinite or indefinite matrices that arise in linear quadratic control problems and differential games−both stochastic and deterministic. The purpose of this paper is to prove some fundamental properties such as comparison, monotonicity and existence theorems. These properties are well-known for classical Riccati differential equations when certain matrices are assumed definite. As applications, we obtain conditions for the existence of solutions to the algebraic Riccati equation and to equations with periodic coefficients.

Keywords: Riccati equation, comparison theorem, upper and lower solution, monotonicity, existence.

1 Introduction

We first state the problem to be considered and then discuss its motivation. Let A, B_i, N, R_{ij} and S_i $(i, j = 1, 2)$ be bounded measurable matrix functions on $[0, T]$ with appropriate dimensions as described below in (6). Let $\mathbf{S}^n$ be the set of all symmetric $n \times n$ matrices. Let $\Pi = (\pi_{ij})_{0 \le i,j \le 2}$ be an operator on $[0, T] \times \mathbf{S}^n$ (see (7)). We consider the following rational matrix equation (or generalized Riccati equation) for a differentiable symmetric matrix function $P(t)$, $0 \le t \le T$:

$$\begin{cases} \dot{P}(t) + \mathcal{N}(t, P(t)) - \mathcal{S}'(t, P(t))\mathcal{R}(t, P(t))^{\dagger}\mathcal{S}(t, P(t)) = 0; & (1.1) \\ P(T) = G \in \mathbf{S}^n; & \\ \text{Range}[\mathcal{S}(t, P(t))] \subset \text{Range}[\mathcal{R}(t, P(t))]; & (1.2) \\ R_{11}(t) + \pi_{11}(t, P(t)) \geq 0;\ R_{22}(t) + \pi_{22}(t, P(t)) \leq 0, & (1.3) \end{cases} \tag{1}$$

where $\dot{P}(t) = \frac{d}{dt}P(t)$, $(\cdot)^{\dagger}$ represents matrix pseudoinverse, $(\cdot)'$ represents matrix transpose, and

$$\begin{aligned} \mathcal{N}(t, P) &= PA'(t) + A(t)P + \pi_{00}(t, P) + N(t); \\ \mathcal{S}(t, P) &= \begin{pmatrix} B_1'(t)P + \pi_{10}(t, P) + S_1(t) \\ B_2'(t)P + \pi_{20}(t, P) + S_2(t) \end{pmatrix}; \\ \mathcal{R}(t, P) &= \begin{pmatrix} R_{11}(t) + \pi_{11}(t, P) & R_{12}(t) + \pi_{12}(t, P) \\ R_{21}(t) + \pi_{21}(t, P) & R_{22}(t) + \pi_{22}(t, P) \end{pmatrix}. \end{aligned} \tag{2}$$

A typical problem where equation (1) arises is a linear quadratic (LQ) stochastic zero-sum game. Specifically, let $\pi_{ij}(t, P) = D_i'(t)PD_j(t)$ for $i = 0, 1, 2$, where D_0, D_1 and D_2 are bounded measurable matrix functions on $[0, T]$ with appropriate dimensions. Then equation (1) is the Riccati equation induced from a stochastic zero-sum game with the following linear state equation and quadratic index.

$$\begin{cases} dx = (Ax + B_1u_1 + B_2u_2)dt + (D_0x + D_1u_1 + D_2u_2)dW(t), \\ \quad \text{for } 0 \leq t \leq T; x(0) = x_0 \\ J(u_1, u_2) = \mathbf{E}[x'(T)Gx(T)] + \mathbf{E}\int_0^T [x'Nx \\ \quad +2x'(S_1'u_1 + S_2'u_2) + u_1'R_{11}u_1 + 2u_1'R_{12}u_2 + u_2'R_{22}u_2]dt \end{cases} \tag{3}$$

where W is a standard Brownian motion over a probability space, $x \in \mathbf{R}^n$ is the state variable, $u_1 \in \mathbf{R}^{k_1}$ and $u_2 \in \mathbf{R}^{k_2}$ are control variables taken by two players, and $\mathbf{E}[\cdot]$ is the expectation; see [1] (for LQ deterministic games) or [2]. In fact, it has recently been proved by the second author and J.M. Yong that equation (1) with $\pi_{ij}(t, P) = D_i'(t)PD_j(t)$ is equivalent to LQ game problems under appropriate conditions. This shows that equation (1) plays a central role in LQ games. It is worth pointing out that while the LQ game is stochastic (with deterministic coefficients), equation (1) is completely deterministic.

Equation (1) is quite general. First, the weighting matrix $\mathcal{R}(t, P)$ is typically indefinite, due to condition (1.3). This is in contrast to Riccati differential equations arising from LQ control problems, where the weighting matrices are necessarily semidefinite. For example, a minimization LQ stochastic control problem can be considered as a special game (3) with u_2 restricted to be 0 (i.e. $k_2 = 0$). In this case, $\mathcal{R}(t, P(t)) = R_{11}(t) + D_1'PD_1$ and a necessary condition for the existence of an optimal control is $\mathcal{R}(t, P(t)) \geq 0$. Note that when $D_1 \neq 0$ (stochastic LQ controls), this condition may be satisfied even if $R_{11}(t)$ (the weight matrix of the control u_1 in the index J) is indefinite; see [3],

[4], [5], [6], [7], and [8]. However, when $D_1 = 0$ (deterministic LQ controls), it is necessary that $\mathcal{R}(t, P(t)) = R_{11}(t) \geq 0$.

In addition, $\mathcal{R}(t, P(t))$ is allowed to be singular here, while most of existing literature assumes that $\mathcal{R}(t, P(t))$ is nonsingular. Finally, with the operator $\Pi(\cdot, \cdot)$, equation (1) contains Riccati equations arising from LQ controls and games with stochastic parameters having jumps; see [9], [10], [11] for such examples from LQ controls.

The purpose of this paper is to establish the fundamental properties for equation (1), including comparison and monotonicity theorems and conditions for existence of solutions. These properties form the basis for solving the equation and the problems where the equation arises. Specifically, in Section 3, we will study the structure of (1) and obtain a useful representation and interpretations for (1). In Section 4, we prove comparison theorems for solutions of (1), generalizing the related results in [10], [11], [12], [13], [14], [15], [16], [17] for special cases of (1). As an application of the comparison theorems, in Section 5 we give a necessary and sufficient condition for the existence of solutions to (1) satisfying a strengthened version of (1.3). As another application of the comparison theorems, we study two interesting special cases in Section 6, where we prove the monotonicity of solutions to (1) in case the coefficients are either constant (time-invariant) or periodic. Consequently, we obtain some necessary and sufficient conditions for the existence of a solution to the algebraic Riccati equation; i.e. a constant solution to (1.1). Finally, we give conditions guaranteeing periodic solutions to (1).

2 Assumptions

In this section we state the assumptions for equation (1). Note that the block matrices in (2) are linear in $P(t)$ and we have

$$\begin{aligned}&\begin{pmatrix} \mathcal{N}(t,P) & \mathcal{S}'(t,P) \\ \mathcal{S}(t,P) & \mathcal{R}(t,P) \end{pmatrix} \\ &= \mathcal{T}(t) + \Pi(t,P) + \begin{pmatrix} A'(t)P + PA(t) & PB_1(t) & PB_2(t) \\ B_1'(t)P & 0 & 0 \\ B_2'(t)P & 0 & 0 \end{pmatrix}\end{aligned} \tag{4}$$

where $\Pi(\cdot, \cdot) = (\pi_{ij}(\cdot, \cdot))_{0 \leq i,j \leq 2}$, and

$$\mathcal{T}(t) = \begin{pmatrix} N(t) & S_1'(t) & S_2'(t) \\ S_1(t) & R_{11}(t) & R_{12}(t) \\ S_2(t) & R_{21}(t) & R_{22}(t) \end{pmatrix}. \tag{5}$$

We will state the assumptions for equation (1) in terms of the quadruple $(A, B, \mathcal{T}, \Pi)$ that determines (1).

Before we continue, let us make some comments on notations. In the rest of paper, we will often suppress the variable t. To indicate that a definition or relationship holds for all $t \in [0, T]$, we will sometimes replace "(t)" by "$(\cdot)$"

and drop the phrase for "for all $t \in [0,T]$". For example, we will write "$P(\cdot) \geq 0$" for "$P(t) \geq 0$ for all $t \in [0,T]$". For a Hilbert space H, we denote by $L^\infty(0,T;H)$ the space of bounded measurable functions with values in H, and by $L^{1,\infty}(0,T;H)$ the space of functions $P \in L^\infty(0,T;H)$ with $\frac{dP}{dt} \in L^\infty(0,T;H)$. The notation $\mathbf{R}^{k\times n}$ will denote the $k \times n$ matrices over the reals.

We make two standing assumptions on the quadruple $(A, B, \mathcal{T}, \Pi)$. Let n, k_1 and k_2 be nonnegative integers.

(A1). The matrix functions $A, B_i, \mathcal{T}$ satisfy

$$\begin{aligned}
&A \in L^\infty(0,T;\mathbf{R}^{n\times n});\\
&B_i \in L^\infty(0,T;\mathbf{R}^{n\times k_i});\ \ S_i \in L^\infty(0,T;\mathbf{R}^{k_i\times n});\\
&R_{ij} = R'_{ji} \in L^\infty(0,T;\,\mathbf{R}^{k_i\times k_j});\\
&N \in L^\infty(0,T;\mathbf{S}^n);\quad G \in \mathbf{S}^n.
\end{aligned} \tag{6}$$

(A2). The operator $\Pi : [0,T] \times \mathbf{S}^n \to \mathbf{S}^{n+k_1+k_2}$ is Lipschitz and monotonically increasing. In other words, there exists a constant L such that for all $t \in [0,T]$ and P, $Q \in \mathbf{S}^n$,

$$\begin{aligned}
&|\Pi(t,P) - \Pi(t,Q)| \leq L|P-Q| \text{ and,}\\
&\text{if } P \leq Q, \text{ then } \Pi(t,P) \leq \Pi(t,Q),
\end{aligned} \tag{7}$$

where $|\cdot|$ is a norm on $\mathbf{S}^n$ or $\mathbf{S}^{n+k_1+k_2}$.

The range hypothesis (1.2) will be used in two slightly different ways. First, $\mathcal{S}(t,P(t))$ and $\mathcal{R}(t,P(t))$ are $(k_1+k_2)\times n$ and $(k_1+k_2)\times(k_1+k_2)$ matrices, respectively. As such they act on vectors in $\mathbf{R}^n$ and $\mathbf{R}^{k_1+k_2}$ both resulting in vectors in $\mathbf{R}^{k_1+k_2}$. (This viewpoint will be used in the proof of Proposition 1.) Second, $\mathcal{S}(t,P(t))$ and $\mathcal{R}(t,P(t))$ can be thought of as acting by multiplication on $n\times(k_1+k_2)$ and $(k_1+k_2)\times(k_1+k_2)$ matrices, respectively, thus resulting in $(k_1+k_2)\times(k_1+k_2)$ matrices. (This will be used in (16).) So hypothesis (1.2) makes sense in either interpretation.

3 Algebraic Descriptions of Riccati Equation (1)

In this section we give a representation and some interpretations for equation (1). The representation is important to the proofs of our main results. We will assume that $(A,B,\mathcal{T})$ satisfies (A1) and Π satisfies (A2), and $P \in L^{1,\infty}(0,T;\mathbf{S}^n)$.

We first recall some properties of the matrix pseudoinverse [18]. For $M \in \mathbf{R}^{k\times n}$, there exists a unique generalized inverse matrix $M^\dagger \in \mathbf{R}^{n\times k}$ that has the following properties:

$$MM^\dagger M = M,\ M^\dagger MM^\dagger = M^\dagger,\ (MM^\dagger)' = MM^\dagger,\ (M^\dagger M)' = M^\dagger M. \tag{8}$$

In addition, if $M = M'$, then

$$M^\dagger = (M^\dagger)',\ M \geq 0 \Leftrightarrow M^\dagger \geq 0,\ M^\dagger M = MM^\dagger.$$

Conditions (1.1)-(1.3) in equation (1) have some algebraic interpretations. Consider the following quadratic form q of $(x, u_1, u_2) \in \mathbf{R}^n \times \mathbf{R}^{k_1} \times \mathbf{R}^{k_2}$.

$$\begin{aligned} q(x, u_1, u_2) &= \begin{pmatrix} x \\ u \end{pmatrix}' \begin{pmatrix} \mathcal{N} & \mathcal{S}' \\ \mathcal{S} & \mathcal{R} \end{pmatrix} \begin{pmatrix} x \\ u \end{pmatrix} \\ &= x'\mathcal{N}x + 2x'\mathcal{S}u + u'\mathcal{R}u, \end{aligned} \tag{9}$$

where $\mathcal{N} \in \mathbf{S}^n$, $\mathcal{R} \in \mathbf{S}^{k_1+k_2}$ and $\mathcal{S} \in \mathbf{R}^{(k_1+k_2)\times n}$ are constant matrices and $u = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$. In the following, we will sometimes use the "row notation" $u = (u_1, u_2)$, for an element of $\mathbf{R}^{k_1} \times \mathbf{R}^{k_2}$, and write $q(x, u_1, u_2) = q(x, u)$. The notation is chosen here to mimic that in equation (1).

Proposition 1 *The following two statements are equivalent.*
(1). $\mathcal{N}$, $\mathcal{R}$ and $\mathcal{S}$ satisfy the conditions

$$\begin{cases} \mathcal{N} - \mathcal{S}'\mathcal{R}^\dagger\mathcal{S} = 0, & \textit{(10.1)} \\ \textit{Range}[\mathcal{S}] \subset \textit{Range}[\mathcal{R}], & \textit{(10.2)} \\ \mathcal{R}_{11} \geq 0, \mathcal{R}_{22} \leq 0. & \textit{(10.3)} \end{cases} \tag{10}$$

(2) For every $x \in \mathbf{R}^n$, the quadratic function $q(x, u_1, u_2)$ has a saddle point $(\widehat{u}_1, \widehat{u}_2) \in \mathbf{R}^{k_1} \times \mathbf{R}^{k_2}$ with value 0, that is, for all $(u_1, u_2) \in \mathbf{R}^{k_1} \times \mathbf{R}^{k_2}$, we have

$$q(x, \widehat{u}_1, u_2) \leq 0 \leq q(x, u_1, \widehat{u}_2). \tag{11}$$

Proof. We first prove (2) $\Rightarrow$ (1). Suppose $(\widehat{u}_1, \widehat{u}_2)$ satisfies (11), then $\widehat{u} = \begin{pmatrix} \widehat{u}_1 \\ \widehat{u}_2 \end{pmatrix}$ is a critical point (depending on x) of $q(x, u_1, u_2)$ in (9). So differentiating (9) with respect to u, we have

$$\mathcal{R}\widehat{u} + \mathcal{S}x = 0. \tag{12}$$

Since x is arbitrary, (10.2) holds. Condition (10.3) follows since (11) holds for all $(u_1, u_2) \in \mathbf{R}^{k_1} \times \mathbf{R}^{k_2}$.

We can rewrite quadratic $q(x, u_1, u_2)$ as a "sum of squares:"

$$q(x, u_1, u_2) = x'(\mathcal{N} - \mathcal{S}'\mathcal{R}^\dagger\mathcal{S})x + (u - \widehat{u})'\mathcal{R}(u - \widehat{u}). \tag{13}$$

Indeed, by using (12): $\mathcal{R}\widehat{u} = -\mathcal{S}x$ and (8): $\mathcal{R} = \mathcal{R}\mathcal{R}^\dagger\mathcal{R}$, we have $(\mathcal{R}\widehat{u})' = -x'\mathcal{S}'$, and so

$$\begin{aligned} (u - \widehat{u})'\mathcal{R}(u - \widehat{u}) =& u'\mathcal{R}u - 2u'\mathcal{R}\widehat{u} + \widehat{u}'\mathcal{R}\widehat{u} \\ =& u'\mathcal{R}u + 2u'\mathcal{S}x + (\mathcal{R}\widehat{u})'\mathcal{R}^\dagger\mathcal{R}\widehat{u} \\ =& u'\mathcal{R}u + 2u'\mathcal{S}x + x'\mathcal{S}'\mathcal{R}^\dagger\mathcal{S}x. \end{aligned}$$

So (13) follows by recalling the definition of $q(x, u_1, u_2)$. By (11) and (13),

$$0 = q(x, \widehat{u}_1, \widehat{u}_2) = x'(\mathcal{N} - \mathcal{S}'\mathcal{R}^\dagger\mathcal{S})x$$

for all x. So (10.1) holds. This finishes the proof of sufficiency.

Now we show that (1) $\Rightarrow$ (2). Condition (10.2) implies that (12) has a solution $\widehat{u} \in \mathbf{R}^{(k_1+k_2)}$ for every $x \in \mathbf{R}^n$. By (13) and (10.1) we have

$$q(x, u_1, u_2) = (u - \widehat{u})'\mathcal{R}(u - \widehat{u}).$$

Then condition (10.3) implies (11).□

Algebraic Interpretations of Riccati Equations. Now we apply Proposition 1 to

$$\begin{pmatrix} \mathcal{N} & \mathcal{S}' \\ \mathcal{S} & \mathcal{R} \end{pmatrix} = \begin{pmatrix} \dot{P}(t) + \mathcal{N}(t, P(t)) & \mathcal{S}'(t, P(t)) \\ \mathcal{S}(t, P(t)) & \mathcal{R}(t, P(t)) \end{pmatrix}, \tag{14}$$

which defines the following quadratic of $u = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \in \mathbf{R}^{k_1+k_2}$ for each $(t, x) \in [0, T] \times \mathbf{R}^n$:

$$x'[\dot{P}(t) + \mathcal{N}(t, P(t))]x + 2x'\mathcal{S}(t, P(t))u + u'\mathcal{R}(t, P(t))u. \tag{15}$$

We obtain the following interpretations of equation (1) for each $(t, x) \in [0, T] \times \mathbf{R}^n$:

(1) Condition (1.2) is equivalent to: quadratic (15) has a critical point $\widehat{u}$.
(2) Condition (1.3) is equivalent to: a critical point of (15) is a saddle point.
(3) Condition (1.1) says that the critical value is 0.

Condition (10.2) is also equivalent to the statement that the matrix equation

$$\mathcal{R}K + \mathcal{S} = 0 \tag{16}$$

has a solution $\widehat{K} \in \mathbf{R}^{(k_1+k_2)\times n}$. In this case, $\widehat{u} = \widehat{K}x$ satisfies (12) for every $x \in \mathbf{R}^n$. In fact, we can show that $-\mathcal{R}^\dagger\mathcal{S}$ is such a solution. Indeed, since $\mathcal{S} = -\mathcal{R}\widehat{K}$ and (8): $\mathcal{R}\mathcal{R}^\dagger\mathcal{R} = \mathcal{R}$, we have $\mathcal{R}\left\{-\mathcal{R}^\dagger\mathcal{S}\right\} = \mathcal{R}\mathcal{R}^\dagger\mathcal{R}\widehat{K} = \mathcal{R}\widehat{K} = -\mathcal{S}$.

Let $K \in \mathbf{R}^{(k_1+k_2)\times n}$ be arbitrary and $u = Kx$. Then

$$q(x, u) = q(x, Kx) = x'[\mathcal{N} + 2K'\mathcal{S} + K'\mathcal{R}K]x.$$

On the other hand, $q(x, Kx)$ is represented by (13) with $\widehat{u} = \widehat{K}x$. Since $x \in \mathbf{R}^n$ is arbitrary, we obtain the following identity for $K \in \mathbf{R}^{(k_1+k_2)\times n}$, which may be verified directly.

$$\begin{aligned} &\mathcal{N}(P) - \mathcal{S}'(P)\mathcal{R}(P)^\dagger\mathcal{S}(P) + (K - \widehat{K})'\mathcal{R}(P)(K - \widehat{K}) \\ &= \mathcal{N} + 2K'\mathcal{S} + K'\mathcal{R}K = \begin{pmatrix} I \\ K \end{pmatrix}' \begin{pmatrix} \mathcal{N} & \mathcal{S}' \\ \mathcal{S} & \mathcal{R} \end{pmatrix} \begin{pmatrix} I \\ K \end{pmatrix}. \end{aligned} \tag{17}$$

Now apply (17) to the matrix in (14) for every $(t, x) \in [0, T] \times \mathbf{R}^n$. The decomposition (4) implies that

$$\begin{aligned} &\begin{pmatrix} I \\ K \end{pmatrix}' \begin{pmatrix} \mathcal{N} & \mathcal{S}' \\ \mathcal{S} & \mathcal{R} \end{pmatrix} \begin{pmatrix} I \\ K \end{pmatrix} = \begin{pmatrix} I \\ K \end{pmatrix}' \begin{pmatrix} \dot{P}(t) + \mathcal{N}(t, P(t)) & \mathcal{S}'(t, P(t)) \\ \mathcal{S}(t, P(t)) & \mathcal{R}(t, P(t)) \end{pmatrix} \begin{pmatrix} I \\ K \end{pmatrix} \\ &= \dot{P}(t) + \mathcal{Q}_0(t, K) + \mathcal{L}(t, K; P(t)) \end{aligned} \tag{18}$$

where $\mathcal{Q}_0(t, K)$ is a quadratic in K and $\mathcal{L}(t, K; P)$ is linear in P, defined as follows.

$$
\begin{aligned}
\mathcal{Q}_0(t, K) &= \begin{pmatrix} I \\ K \end{pmatrix}' \mathcal{T}(t) \begin{pmatrix} I \\ K \end{pmatrix}, \\
\mathcal{L}(t, K; P) &= \begin{pmatrix} I \\ K \end{pmatrix}' \Pi(t, P) \begin{pmatrix} I \\ K \end{pmatrix} + [A(t) + B(t)K]' P + P [A(t) + B(t)K],
\end{aligned} \tag{19}
$$

where $B = (B_1, B_2)$ and $\mathcal{T}$ is defined in (5). From (17) and (18), we obtain the following representations of (1).

Proposition 2 *Suppose for each* $t \in [0, T]$*,* $P(t) \in L^{1,\infty}(0, T; \mathbf{S}^n)$ *satisfies (1.2). Then for each* $K \in \mathbf{R}^{(k_1+k_2)\times n}$*, letting* $\mathcal{K}(t, P) = -\mathcal{R}^\dagger(t, P)\mathcal{S}(t, P)$*, we have*

$$
\begin{aligned}
&\dot{P}(t) + \mathcal{N}(t, P(t)) - \mathcal{S}'(t, P(t))\mathcal{R}(t, P(t))^\dagger \mathcal{S}(t, P(t)) \\
&= \dot{P}(t) + \mathcal{Q}_0(t, K) + \mathcal{L}(t, K; P) \\
&\quad - (K - \mathcal{K}(t, P(t)))' \mathcal{R}(t, P(t))(K - \mathcal{K}(t, P(t))).
\end{aligned} \tag{20}
$$

4 Comparison Theorems

Comparison theorems are a fundamental part of the theory of Riccati equations. Such theorems are a basis for proving the existence of solutions and other properties of Riccati equations. For special cases of equation (1) (as mentioned in §1), comparison theorems have been proved in many papers. See [10], [11], [13], [14], [15], [17], [19], [20], [21] and references therein for more exposition on comparison theorems and some of their applications. We prove two comparison theorems (essentially equivalent) for equation (1).

First consider an equation of the form

$$
\dot{P}(t) + A'(t)P(t) + P(t)A(t) + \pi(t, P(t)) + N(t) = 0, \quad P(T) = G, \tag{21}
$$

where the operator $\pi(\cdot, \cdot) : [0, T] \times \mathbf{S}^n \to \mathbf{S}^n$. We will write $\pi(\cdot, \cdot) \geq 0$ to mean that $\pi(t, P) \geq 0$ for all $(t, P) \in [0, T] \times \mathbf{S}^n$ with $P \geq 0$.

Proposition 3 *Let* A, G, N *be as in (6) and* $\pi(\cdot, \cdot)$ *be Lipschitz. Then equation (21) has a unique solution* $P \in L^{1,\infty}(0, T; \mathbf{S}^n)$*. In addition, if* $\pi(\cdot, \cdot) \geq 0$*,* $G \geq 0$ *and* $N(\cdot) \geq 0$*, then* $P(\cdot) \geq 0$*. Furthermore, if* $\pi(\cdot, \cdot) \geq 0$*, and either* $G > 0$ *or* $N(\cdot) > 0$*, then* $P(\cdot) > 0$*.*

Proof. Following the idea in [17] we let $\Phi(t, s)$ be the fundamental matrix of A, that is, $\Phi(t, s)^{-1} = \Phi(s, t)$ and $\frac{\partial}{\partial t}\Phi(s, t) = -A(t)\Phi(s, t)$. It follows that (21) is equivalent to

$$
P(t) = \Phi(T, t)' G \Phi(T, t) + \int_t^T \Phi(s, t)' [\pi(s, P(s)) + N(s)] \Phi(s, t) ds. \tag{22}
$$

The Volterra equation (22) has a unique solution P, which can be found by successive approximations; say, $\{P_m : m = 0, 1, 2, \dots\}$ starting with $P_0 = 0$. If $\pi(\cdot, \cdot) \geq 0$, $G \geq 0$ and $N(\cdot) \geq 0$, then $P_m(\cdot) \geq 0$ for all $m \geq 0$, which implies that $P(\cdot) = \lim_{m\to\infty} P_m(\cdot) \geq 0$. If either $G > 0$ or $N(\cdot) > 0$, then for all $t \in [0, T]$,

$$P(t) \geq \Phi(T,t)'N\Phi(T,t) + \int_t^T \Phi(s,t)'G(s)\Phi(s,t)ds > 0. \quad \square$$

Theorem 1 (Comparison) *For $i = 1, 2$, let $P_i \in L^{1,\infty}(0, T; \mathbf{S}^n)$ be solutions to equation (1) associated $(A, B, \mathcal{T}_i, \Pi_i)$, which satisfy (A1) and (A2). Suppose $\Pi_1(\cdot, \cdot) \leq \Pi_2(\cdot, \cdot)$, $\mathcal{T}_1(\cdot) \leq \mathcal{T}_2(\cdot)$ and $P_1(T) \leq P_2(T)$. Then $P_1(\cdot) \leq P_2(\cdot)$. In addition, $P_1(\cdot) < P_2(\cdot)$ if one of the following conditions holds.*

(1) $\mathcal{T}_2 - \mathcal{T}_1 = \begin{pmatrix} N_2 - N_1 & 0 \\ 0 & 0 \end{pmatrix}$ (i.e., $\mathcal{T}_2$ and $\mathcal{T}_1$ are different at only the $(1,1)$ entry) and $N_2(\cdot) - N_1(\cdot) > 0$.

(2) $P_1(T) < P_2(T)$.

Proof. For $i = 1, 2$, let $\mathcal{S}_i(P_i)$, $\mathcal{R}_i(P_i)$, $\mathcal{K}_i(P_i)$, $\mathcal{Q}_{i0}(K), \mathcal{L}_i(K; P_i)$ be the notations defined in (2), Proposition 2 and (19) with $(A, B, \mathcal{T}_i, \Pi_i)$ (with t suppressed). Using representation (20) for the equations of P_1 and P_2, we obtain for arbitrary $K \in \mathbf{R}^{(k_1+k_2)\times n}$,

$$0 = \mathcal{Q}_{20}(K) - \mathcal{Q}_{10}(K) + \dot{P}_2 - \dot{P}_1 + \mathcal{L}_2(K; P_2) - \mathcal{L}_1(K; P_1) + M(K) \qquad (23)$$

where $M(K)$ is the difference of "square" terms in (20), that is

$$\begin{aligned} M(K) =& (K - \mathcal{K}_1(P_1))'\mathcal{R}_1(P_1)(K - \mathcal{K}_1(P_1)) \\ &- (K - \mathcal{K}_2(P_2))'\mathcal{R}_2(P_2)(K - \mathcal{K}_2(P_2)). \end{aligned}$$

A key step of the proof is to choose K such that $M(K) \geq 0$. Write $\mathcal{K}_1(P_1) = \begin{pmatrix} \mathcal{K}_{11}(P_1) \\ \mathcal{K}_{12}(P_1) \end{pmatrix}$ and $\mathcal{K}_2(P_2) = \begin{pmatrix} \mathcal{K}_{21}(P_2) \\ \mathcal{K}_{22}(P_2) \end{pmatrix}$ and choose $K = \begin{pmatrix} \mathcal{K}_{21}(P_2) \\ \mathcal{K}_{12}(P_1) \end{pmatrix}$. It follows that

$$K - \mathcal{K}_1(P_1) = \begin{pmatrix} \mathcal{K}_{21}(P_2) \\ \mathcal{K}_{12}(P_1) \end{pmatrix} - \begin{pmatrix} \mathcal{K}_{11}(P_1) \\ \mathcal{K}_{12}(P_1) \end{pmatrix} = \begin{pmatrix} \mathcal{K}_{21}(P_2) - \mathcal{K}_{11}(P_1) \\ 0 \end{pmatrix}.$$

Similarly $K - \mathcal{K}_2(P_2) = \begin{pmatrix} 0 \\ \mathcal{K}_{12}(P_1) - \mathcal{K}_{21}(P_2) \end{pmatrix}$. Using the notation $[\cdot]_{i,j}$ to denote the (i,j) entry of a matrix, we know from condition (1.3) that $[\mathcal{R}_1(P_1)]_{1,1} \geq 0$ and $[\mathcal{R}_2(P_2)]_{2,2} \leq 0$. It follows that

$$\begin{aligned} M(K) =& (\mathcal{K}_{21}(P_2) - \mathcal{K}_{11}(P_1))'[\mathcal{R}_1(P_1)]_{1,1}(\mathcal{K}_{21}(P_2) - \mathcal{K}_{11}(P_1)) \\ &- (\mathcal{K}_{12}(P_1) - \mathcal{K}_{22}(P_2))'[\mathcal{R}_2(P_2)]_{2,2}(\mathcal{K}_{12}(P_1) - \mathcal{K}_{22}(P_2)) \geq 0. \end{aligned}$$

Denote $P(\cdot) = P_2(\cdot) - P_1(\cdot)$. Then $P(T) \geq 0$ and (23) implies that P satisfies

$$\begin{aligned} 0 =& \begin{pmatrix} I \\ K \end{pmatrix}' [\mathcal{T}_2 - \mathcal{T}_1 + \Pi_2(P + P_1) - \Pi_1(P_1)] \begin{pmatrix} I \\ K \end{pmatrix} \qquad (24) \\ &+ \dot{P} + (A + BK)'P + P(A + BK) + M(K). \end{aligned}$$

Now we apply Proposition 3 to (24) with $N = M(K)+\begin{pmatrix} I \\ K \end{pmatrix}' [\mathcal{T}_2 - \mathcal{T}_1] \begin{pmatrix} I \\ K \end{pmatrix}$ and $\Pi(\cdot) = \Pi_2(\cdot + P_1) - \Pi_1(P_1)$. Since $M(K) \geq 0$ and $\mathcal{T}_2(\cdot) \geq \mathcal{T}_1(\cdot)$, we have $N(\cdot) \geq 0$. In addition, since $\Pi_2(\cdot,\cdot) \geq \Pi_1(\cdot,\cdot)$ and Π_1 is monotonically increasing, we have, for every $Q \geq 0$,

$$\Pi(t, Q) = \Pi_2(t, Q + P_1) - \Pi_1(t, P_1) \geq \Pi_1(t, Q + P_1) - \Pi_1(t, P_1) \geq 0.$$

So $\Pi(\cdot,\cdot) \geq 0$. By Proposition 3, $P(\cdot) \geq 0$.

Under condition (1), we have for $t \in [0, T]$,

$$N = N_2 - N_1 + M(K) \geq N_2 - N_1 > 0.$$

So under either condition (1) or (2), $P(\cdot) = P_2(\cdot) - P_1(\cdot) > 0$ by Proposition 3. □

The comparison theorem is often stated in the terms of upper and lower solutions. We say that $P \in L^{1,\infty}(0, T; \mathbf{S}^n)$ is a *lower* [*upper*, respectively] solution of (1) if it satisfies conditions (1.2) and (1.3) and the following inequalities for all $t \in [0, T]$:

$$\begin{aligned}&\dot{P}(t) + \mathcal{N}(t, P(t)) - \mathcal{S}'(t, P(t))\mathcal{R}(t, P(t))^\dagger \mathcal{S}(t, P(t)) \geq 0;\ P(T) \leq G.\\ &\text{[and, respectively,}\\ &\dot{P}(t) + \mathcal{N}(t, P(t)) - \mathcal{S}'(t, P(t))\mathcal{R}(t, P(t))^\dagger \mathcal{S}(t, P(t)) \leq 0;\ P(T) \geq G]\end{aligned} \tag{25}$$

The terminology may seem improperly named, but it is simply a reflection of the fact that the problems being solved are final value rather than initial value problems. Finally, if one of the inequalities in (25) is strict, then we say P is a *strict* lower (or upper) solution.

For example, it is easy to see that $P(\cdot) = 0$ is a lower [an upper] solution of (1) if and only if for all $t \in [0, T]$,

$$\begin{aligned}&R_{11}(t) \geq 0, R_{22}(t) \leq 0, \text{Range}[S(t)] \subset \text{Range}[R(t)];\\ &N(t) - S'(t)R^\dagger(t)S(t) \geq 0,\ G \geq 0.\\ &[N(t) - S'(t)R^\dagger(t)S(t) \leq 0,\ G \leq 0, \text{respectively.}]\end{aligned} \tag{26}$$

Using Theorem 1, we now prove the following comparison result.

Theorem 2 (Comparison) *Suppose* (P_1, P_2) *is a pair of lower-upper solutions of (1). Then* $P_1(\cdot) \leq P_2(\cdot)$. *If either* P_1 *is a strict lower, solution or* P_2 *is a strict upper solution, then* $P_1(\cdot) < P_2(\cdot)$.

Proof. The assumption implies that

$$\dot{P}_i(t) + \mathcal{N}(t, P_i(t)) - \mathcal{S}'(t, P_i(t))\mathcal{R}(t, P_i(t))^\dagger \mathcal{S}(t, P_i(t)) + H_i(t) = 0.$$

for some matrices $H_1(\cdot) \leq 0$, $H_2(\cdot) \geq 0$ and for all $t \in [0, T]$. Apply Theorem 1 with $\mathcal{T}_2 = \begin{pmatrix} N + H_2 & S' \\ S & R \end{pmatrix}$ and $\mathcal{T}_1 = \begin{pmatrix} N + H_1 & S' \\ S & R \end{pmatrix}$. Since $\mathcal{T}_2 \geq \mathcal{T}_1$ and

$P_1(T) \leq G \leq P_2(T)$, $P_1(\cdot) \leq P_2(\cdot)$ by Theorem 1. The second conclusion follows from the second conclusion of Theorem 1. $\square$

As a corollary, we obtain the following result.

Corollary 1 *Suppose condition (26) holds. Then the solution P of (1), if it exists, is positive [negative, respectively] semidefinite.*

5 Existence Results

Upper and lower solutions can usually be found more easily than solutions. In this section we give a condition for the existence of a solution to equation (1) in terms of upper and lower solutions. We consider solutions $P \in L^{1,\infty}(0,T;\mathbf{S}^n)$ that satisfy the strict inequalities in (1.3), that is, for $t \in [0,T]$,

$$\begin{cases} \dot{P}(t) + \mathcal{N}(t,P(t)) - \mathcal{S}'(t,P(t))\mathcal{R}(t,P(t))^{\dagger}\mathcal{S}(t,P(t)) = 0, & (27.1) \\ P(T) = G \in \mathbf{S}^n, \\ R_{11}(t) + \pi_{11}(t,P(t)) > 0;\ R_{22}(t) + \pi_{22}(t,P(t)) < 0. & (27.2) \end{cases} \quad (27)$$

We assert first that if P satisfies (27.2), then the weighting matrix $\mathcal{R}(P(t))$ is invertible. The invertibility of $\mathcal{R}(t,P(t))$ can be seen by writing its block matrix form, for convenience, as $\mathcal{R}(P(t)) = \begin{pmatrix} A & B \\ B' & C \end{pmatrix}$, with A and C being square matrices, $A > 0$, and $C < 0$. If $\begin{pmatrix} x \\ y \end{pmatrix}$ is in the kernel of $\mathcal{R}(P(t))$, then $Ax + By = 0$ so that $x = -A^{-1}By$. Then $B'x + Cy = 0$ can be rewritten as $(C - B'A^{-1}B)y = 0$. But $C - B'A^{-1}B < 0$, so $y = 0$ and it follows that $\begin{pmatrix} x \\ y \end{pmatrix} = 0$. Thus $\mathcal{R}(t,P(t))$ is invertible which then, of course, implies $\mathcal{R}(t,P(t))^{-1} = \mathcal{R}(t,P(t))^{\dagger}$. In this case, conditions (1.2) and (1.3) are automatically satisfied.

The notion of lower and upper solutions for (27) is defined as in (25). We have

Theorem 3 *There exists a solution $P \in L^{1,\infty}(0,T;\mathbf{S}^n)$ to (27) if and only if (27) has a pair of lower-upper solutions (P_1,P_2).*

Proof. The necessity is trivial by taking $P = P_1 = P_2$. We prove the sufficiency. Since $P_1(T) \leq G \leq P_2(T)$ and P_1 and P_2 satisfy (27.2), it follows that

$$\begin{aligned} R_{11}(T) + \pi_{11}(G) &\geq R_{11}(T) + \pi_{11}(P_1(T)) > 0, \\ R_{22}(T) + \pi_{22}(G) &\leq R_{22}(T) + \pi_{22}(P_2(T)) < 0. \end{aligned}$$

So (27.2) holds at $t = T$ with $P(T) = G$. The local existence theory of ODE implies that equation (27.1) has a solution P that exists in a maximal interval $(\tau, T]$. Theorem 2 implies that $P_1(t) \leq P(t) \leq P_2(t)$ in $(\tau, T]$. This implies that

P satisfies (27.2). By using the inequalities in (27.2) and equation (27.1), we see that $P_\tau \equiv \lim_{t\to\tau+} P(t)$ exists and $P_1(\tau) \le P_\tau \le P_2(\tau)$. It turns out that (27.2) holds for $P(t)$ with $t \in [\tau, T]$. So P can be extended to a solution of (27.1) that satisfies (27.2) on $[\tau, T]$. We claim that $\tau = 0$. For if $\tau > 0$, then P could be extended further left beyond τ by the local existence theory of ODE. This would contradict the maximality of $(\tau, T]$. Therefore equation (27.1) has a solution on $[0, T]$ satisfying (27.2). □

If the lower solution P_1 and upper solution P_2 satisfy (1.2) and (1.3), instead of (27.2), then it is an open question whether equation (1) has a solution. In the proof of Theorem 3 we used the following three properties of solutions to (1) that satisfy (27.2).

(a) If P_1 and P_2 satisfy (27.2), then every function between P_1 and P_2 also satisfies (27.2).

(b) The local existence of a solution of (27).

(c) Every solution of (27) on an interval $(\tau', \tau] \subset [0, T]$ can be extended to $[\tau', \tau]$.

For equation (1.1) with (1.2) and (1.3), (a) can be verified under appropriate kernel conditions, as done in [15] for rational differential equations arising from stochastic controls. Properties (b) and (c) are not known due to the fact that the generalized inverse $\mathcal{R}(t, P(t))^\dagger$ is not continuous in its argument $P(t)$. For this reason, the existence result Corollary 4.6 in [15] may need additional assumptions. We hope to address this question in a future study.

We end this section with the following property of solutions to (27).

Proposition 4 *If (P_1, P_2) is a pair of lower and upper solutions of (27), then the set of solutions, X, of (27) satisfying $P_1 \le X \le P_2$, contains a minimal solution Z and a maximal solution Y. That is,*

$$Z(\cdot) \le X(\cdot) \le Y(\cdot)$$

for all such solutions X.

Proof. Let Y and Z be the solutions with $Y(T) = P_2(T)$ and $Z(T) = P_1(T)$. Now if X is any solution of (27) satisfying $P_1(T) \le X(T) \le P_2(T)$, then we have $Z(\cdot) \le X(\cdot) \le Y(\cdot)$ by Theorem 2. □

6 Monotonicity and existence of constant solutions

In this section we study the monotonicity of a solution of (1) when all coefficient matrices are periodic or constant in t. As applications, we give conditions for the existence of periodic or constant solutions to (1).

When all coefficient matrices are constant, the algebraic Riccati equation for $P \in \mathbf{S}^n$ associated with (1) is

$$\begin{cases} \mathcal{E}(P) \equiv \mathcal{N}(P) - \mathcal{S}'(P)\mathcal{R}(P)^{\dagger}\mathcal{S}(P) = 0; & (28.1) \\ \text{Range}\,[\mathcal{S}(P)] \subset \text{Range}[\mathcal{R}(P)]; & (28.2) \\ R_{11} + \pi_{11}(P) \geq 0;\ R_{22} + \pi_{22}(P) \leq 0, & (28.3) \end{cases} \tag{28}$$

where in analogy with (2)

$$\mathcal{N}(P) = PA' + AP + \pi_{00}(P) + N;$$
$$\mathcal{S}(P) = \begin{pmatrix} B_1'P + \pi_{10}(P) + S_1 \\ B_2'P + \pi_{20}(P) + S_2 \end{pmatrix};$$
$$\mathcal{R}(P) = \begin{pmatrix} R_{11} + \pi_{11}(P) & R_{12} + \pi_{12}(P) \\ R_{21} + \pi_{21}(P) & R_{22} + \pi_{22}(P) \end{pmatrix}.$$

Theorem 4 (Monotonicity) *Suppose all of the coefficients in (1) are constant and $P(t)$ is the solution of (1) in $(s, T]$ with $G = P(T)$. Suppose that*

$$\text{Range}[\mathcal{S}(P)] \subset \text{Range}[\mathcal{R}(P)],$$
$$R_{11} + \pi_{11}(G) \geq 0; R_{22} + \pi_{22}(G) \leq 0.$$

Then we have
(i) $\mathcal{E}(G) \geq 0$ if and only if $P(t)$ is decreasing in $(s, T]$.
(ii) $\mathcal{E}(G) \leq 0$ if and only if $P(t)$ is increasing in $(s, T]$.

Proof. (i) Since $\mathcal{E}(G) \geq 0$, G is a constant lower solution to (1) on $(-\infty, T]$. By Theorem 2, $P(t) \geq G$, for all $t \in (s, T]$. For any number $\tau \in (0, T - s)$, define $P_* : (s + \tau, T + \tau] \to \mathbf{S}^n$ by $P_*(t) = P(t - \tau)$. Since (1) is autonomous, $P_*(t)$ is also a solution to (1) with $P_*(T) = P(T - \tau) \geq G = P(T)$. By Theorem 2 again, $P_*(t) \geq P(t)$ for $t \in (s + \tau, T]$, or equivalently, $P(t - \tau) \geq P(t)$ for every $\tau \in (0, T - s)$. In other words, $P(t)$ is decreasing in $(s, T]$ as a function of t. Part (ii) is proved similarly. □

Extending terminology to the algebraic case, we will call $P \in \mathbf{S}^n$ a lower solution to the algebraic Riccati equation (28) in case $\mathcal{E}(P) \geq 0$ and an upper solution in case $\mathcal{E}(P) \leq 0$. Using this terminology, Theorem 4 can be rephrased by saying $P(\cdot)$ is decreasing if and only if the terminal value G is a lower solution. And similarly $P(\cdot)$ is increasing if and only if the terminal value G is an upper solution.

Theorem 4 implies that if P is a bounded solution to (1) on $(-\infty, T]$ with $G = P(T)$ being an upper or lower solution to (1), then $P_\infty \equiv \lim_{t\to-\infty} P(t)$ exists. By combining Theorem 3 and Theorem 4, we obtain a necessary and sufficient condition for existence of solutions to the following equation:

$$\begin{cases} \mathcal{E}(P) \equiv \mathcal{N}(P) - \mathcal{S}'(P)\mathcal{R}(P)^{\dagger}\mathcal{S}(P) = 0; & (29.1) \\ R_{11} + \pi_{11}(P) > 0;\ R_{22} + \pi_{22}(P) < 0, & (29.2) \end{cases} \tag{29}$$

Note again that (29.2) is a stronger assumption than (28.2) and (28.3).

Theorem 5 *Equation (29) has a solution $P \in \mathbf{S}^n$ if and only if it has an upper solution Y and a lower solution Z such that $Y \geq Z$. The set $[Z, Y] = \{P \in \mathbf{S}^n \mid Z \leq P \leq Y\}$ contains a minimal solution and maximal solution.*

Proof. The necessity is obvious by choosing $Y = Z = P$. For the sufficiency, consider equation (27) with terminal values $P(T) = Y$ and Z, respectively. Since Y is an upper solution and Z is a lower solution to (27) in $(-\infty, T]$, by Theorem 3, there exist solutions $P_Y(\cdot)$ and $P_Z(\cdot)$ on $(-\infty, T]$ to (27) such that $P_Y(T) = Y$, $P_Z(T) = Z$ and $Y \geq P_Y(\cdot) \geq P_Z(\cdot) \geq Z$. By Theorem 4, both $P_Y(\cdot)$ and $P_Z(\cdot)$ are monotone. So both $Y_\infty = \lim_{t\to-\infty} P_Y(t)$ and $Z_\infty = \lim_{t\to-\infty} P_Z(t)$ exist. Clearly Y_∞ and Z_∞ are constant solutions to (27) satisfying $Y \geq Y_\infty \geq Z_\infty \geq Z$. It follows that

$$\begin{aligned}
&R_{11} + \pi_{11}(Y_\infty) \geq R_{11} + \pi_{11}(Z_\infty) \geq R_{11} + \pi_{11}(Z) > 0, \\
&R_{22} + \pi_{22}(Z_\infty) \leq R_{11} + \pi_{11}(Y_\infty) \leq R_{11} + \pi_{11}(Y) < 0. \quad \square
\end{aligned}$$

Lastly consider the case in which all the coefficients of (1.1) are periodic with period θ.

Theorem 6 *Suppose the coefficients of (1.1) are continuous and θ-periodic. Equation (27) has a θ-periodic solution P if and only if (27) has an upper solution Y and a lower solution Z on $[T-\theta, T]$ such that*

$$Z(T) \leq Z(T-\theta) \leq Y(T-\theta) \leq Y(T).$$

Proof. The necessity is obvious by letting $Y = Z = P$. To show the sufficiency, we first show that for $k = 0, 1, 2, \ldots$, (27) has a solution P_i on $[T-\theta, T]$ with the following properties, for $i = 1, 2, \ldots$,

$$\begin{aligned}
&P_0(T) = Z(T), P_{i+1}(T) = P_i(T-\theta), \\
&Z(t) \leq P_i(t) \leq P_{i+1}(t) \leq Y(t) \text{ for } t \in [T-\theta, T].
\end{aligned}$$

The existence of a solution $P_0(t)$ with $Z(t) \leq P_0(t) \leq Y(t)$ for $t \in [T-\theta, T]$ is guaranteed by Theorem 3. Since

$$Z(T) \leq Z(T-\theta) \leq P_0(T-\theta) \leq Y(T-\theta) \leq Y(T),$$

the existence of $P_1(t)$ with $Z(t) \leq P_1(t) \leq Y(t)$ for $t \in [T-\theta, T]$ is guaranteed, again by Theorem 3. Since both $P_1(t)$ and $P_0(t)$ are solutions to (27) with $P_1(T) = P_0(T-\theta) \geq P_0(T)$, the comparison theorem implies that $P_0(t) \leq P_1(t)$. The cases for $i \geq 1$ are proved by induction.

So $\{P_i(\cdot)\}$ is a monotonically increasing bounded sequence. Therefore,

$$P_\infty(t) = \lim_{i\to\infty} P_i(t) \text{ exists for each } t \in [T-\theta, T].$$

The property $P_{i+1}(T) = P_i(T-\theta)$ implies that $P_\infty(T) = P_\infty(T-\theta)$, that is, $P(t)$ is θ-periodic. Since each $P_i(t)$ satisfies (27) and $Z(t) \leq P_i(t) \leq Y(t)$

for $t \in [T-\theta, T]$, the limit $P_\infty(t)$ satisfies (27) and $Z(t) \le P_\infty(t) \le Y(t)$ for $t \in [T-\theta, T]$. Extending $P_\infty(\cdot)$ to $(-\infty, T]$ periodically, we get a periodic solution of (27).

It is a typical argument to show that $P_\infty(\cdot)$ satisfies equation (27). Equation (27) for $P_i(\cdot)$ can be written as $\frac{d}{dt}P_i(t) = -\mathcal{E}(t, P_i(t))$, where

$$\mathcal{E}(t,P) = \mathcal{N}(t,P) - \mathcal{S}'(t,P)\mathcal{R}(t,P)^{-1}\mathcal{S}(t,P).$$

We show that $\mathcal{E}(t, P_i(t))$ is a bounded function uniformly for $i \ge 0$. Write,

$$\mathcal{R}(t, P_i(t)) = (R_{jk}(t) + \pi_{jk}(P_i(t)))_{1\le j,k\le 2} = \begin{pmatrix} A & B \\ B' & C \end{pmatrix}.$$

where

$$A = R_{11}(t) + \pi_{11}(t, P_i(t)) \ge R_{11}(t) + \pi_{11}(t, Z(t))$$
$$C = R_{22}(t) + \pi_{22}(t, P_i(t)) \le R_{22}(t) + \pi_{22}(t, Y(t))$$

Since Y and Z satisfy (27.2) and the coefficients are continuous, we see that A^{-1} and C^{-1} (and hence $(C - B'A^{-1}B)^{-1}$) are bounded functions uniformly for $i \ge 0$. By the inverse formula

$$\begin{pmatrix} A & B \\ B' & C \end{pmatrix}^{-1} = \begin{pmatrix} I & 0 \\ -B'A^{-1} & I \end{pmatrix} \begin{pmatrix} A^{-1} & 0 \\ 0 & (C - B'A^{-1}B)^{-1} \end{pmatrix} \begin{pmatrix} I & -A^{-1}B \\ 0 & I \end{pmatrix}$$

we see that $\mathcal{R}(t, P_i(t))^{-1}$ is bounded uniformly for $i \ge 0$. Consequently we know that $\mathcal{E}(t, P_i(t))$ is a bounded function uniformly for $i \ge 0$. Therefore for all r and $r + \Delta r \in [T-\theta, T]$, we have

$$P_i(t)|_r^{r+\Delta r} = -\int_r^{r+\Delta r} \mathcal{E}(t, P_i(t))dt.$$

Letting $i \to \infty$, we have that

$$P_\infty(t)|_r^{r+\Delta r} = -\int_r^{r+\Delta r} \mathcal{E}(t, P_\infty(t))dt$$

Dividing both sides by Δr and taking limit as $\Delta r \to 0$, we see that $P_\infty(t)$ satisfies (27) on $[T-\theta, T]$. □

Note that the solution $P_\infty(t)$ obtained in the proof of Theorem 6 is the minimal θ-periodic solution between Z and Y on $[T-\theta, T]$. If we start with $P_0(T) = Y(T)$, then the solution $P_\infty(\cdot)$ will be the maximal θ-periodic solution between Z and Y on $[T-\theta, T]$.

In [14, Theorems 3.9 & 3.11], results similar to Theorem 6 are proved for equation (1) among periodic solutions $P(t)$ such that $\mathcal{R}(t, P(t)) > 0$.

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