Bubbling Phenomena of Certain Palais-Smale Sequences of m-Harmonic Type Systems

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Abstract. In this paper, we study the bubbling phenomena of weak solution sequences of a class of degenerate quasilinear elliptic systems of m-harmonic type. We prove that, under appropriate conditions, the energy is preserved during the bubbling process. The results apply to m-harmonic maps from a manifold Ω^m to a homogeneous space, and to m-harmonic maps with constant volumes, and also to certain Palais-Smale sequences.

§1 **Introduction and Main Results**

The Palais-Smale (P-S) condition for a functional E is a natural assumption for the existence of a critical point of E; this condition says that any Palais-Smale sequence $\{u_n\}$ (i.e., $\sup_n |E(u_n)| < \infty$ and $||DE(u_n)|| \to 0$, as $n \to \infty$) has a (strongly) convergent subsequence. In many interesting cases where Palais-Smale condition fails, people discovered the bubbling phenomena of certain Palais-Smale sequences. Generally speaking, the failure of strong convergence is due to the loss of energy, and a bubbling phenomenon refers that the lost energy was recovered (or captured) by a few bubbles (solutions of the blow-up equation) developed during the limit process (or bubbling process). For references on bubbling phenomena and related problems, see [SaU] [Jj] [Sm1] [Pt] [Qj] on harmonic maps on surfaces; [BN] [Sm2] on semilinear elliptic equations, and [Wh] [BC] [Sm3] on H-systems, which describe surfaces of constant mean curvatures.

Suppose (Ω^m, g) is a Riemannian manifold. The space $W^{1,m}\left(\Omega, R^k\right)$ consists of all functions $u : \Omega \to R^k$ with finite energy:

$$E_m(u) = \int_\Omega |Du|^m \, d\Omega,$$

where $d\Omega$ denotes the volume element of Ω, often being omitted. $Du = \left(Du^1, ..., Du^k\right)$ is the differential of u. We assume that $m, k \geq 2$ integers, and $\partial\Omega = \emptyset$.

We are interested in bubbling phenomena of minimizing sequences and Palais-Smale sequences of the energy E subject to certain constraints. The Euler-Lagrange equation can often be written as

$$-\text{div}\left(|Du|^{m-2} Du\right) = f(u, Du), \tag{1.1}$$

where $u \in W^{1,m}\left(\Omega, R^k\right)$. We assume the following hypotheses.

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(H-I). $f : R^k \times R^{mk} \to R^k$ is a smooth function that can be written as:

$$f(u, Du) = \sum_{\alpha=1}^{q} a_\alpha(u, Du) \cdot Db_\alpha(u), \tag{1.2}$$

where $a_\alpha(\cdot,\cdot) : R^k \times R^{mk} \to R^{mk^2}$ and $b_\alpha(\cdot) : R^k \to R^{k^2}$ are smooth vector-valued functions such that

$$|a(u, Du)| \le C|Du|^{m-1}, \ \ |b(u)| \le C$$

for some constant C. Here C may depend on $\|u\|_\infty$. Componentwise, (1.2) becomes

$$f^i(u, Du) = \sum_{j=1}^{k} \sum_{\alpha=1}^{q} \sum_{l=1}^{m} a^{il}_{\alpha j}(u, Du) \cdot \frac{\partial}{\partial x^l} b^i_{\alpha j}(u), \quad i = 1, ..., k.$$

This assumption will be used, together with others, to show that $f(u, Du)$ is in the local Hardy space, better than L^1. In a few interesting cases, this assumption naturally holds; see [Wh1] [Hf] [El] [MY] [TW], and the examples below.

Equation (1.1) is understood in the weak sense: For all $\phi \in C^1_0(\Omega, R^k)$, there holds

$$\int_\Omega |Du|^{n-2} Du D\phi = \int_\Omega f(u, Du)\, \phi. \tag{1.3}$$

We assume that (1.1) is conformally invariant in the following sense.

(H-II) Suppose $\Phi : (\Omega_1^m, h) \to (\Omega^m, g)$ is a conformal diffeomorphism, then (1.3) holds with u replaced by $u \circ \Phi$, ϕ by $\phi \circ \Phi$ and Ω by Ω_1^m.

Note that, since the energy $\int_\Omega |Du|^m$ is conformally invariant, the Euler-Lagrange equation of $\int_\Omega |Du|^m$ subject to a constraint is conformally invariant in the sense of (H-II), as long as the constraint is closed under conformal transformations. The m-harmonic map equations and the H-systems, defined below, are two such examples. This property implies that the equation keeps the same form under conformal transformations, especially under dilations and translations.

We consider a sequence $\{u_n\}_{n=1}^\infty \subset W^{1,m}(\Omega, R^k)$ satisfying perturbed equations of (1.1):

$$-\mathrm{div}\left(|Du_n|^{m-2} Du_n\right) = f(u_n, Du_n) + h_n, \ \ h_n \to 0 \text{ in } W^{-1,m'}, \tag{1.4}$$

for $m' = \frac{m}{m-1}$, and

$$u_n \rightharpoonup u \text{ in } W^{1,m}, \text{ but } u \not\to u \text{ in } W^{1,m}.$$

The study of the convergence behavior of $\{u_n\}$ leads to the below-up equation:

$$-\mathrm{div}\left(|Du|^{m-2} Du\right) = f(u, Du), \tag{1.5}$$

for $u \in W^{1,m}(R^m, R^k)$. This equation is obtained as a limit of the equations satisfied by properly rescaled $\{u_n\}$ (see (4.5)). Here R^m plays the role of tangent spaces of Ω. A nontrivial solution u of

(1.5) is called a *bubble.* By the conformal invariance of the equation (1.5) and the conformal equivalence of R^m and S^m under the stereographic projection, a bubble u is identified with $\tilde{u} \in W^{1,m}\left(S^m, R^k\right)$, which satisfies (1.5) on S^m. Suppose u is regular (say C^1; this is the case if $\operatorname{div}(a\left(u, Du\right)) = 0$, by Thm 2.6 and Prop 3.1 in [MY] , then $\tilde{u}$ is regular on S^m except at the north pole (of the projection). Suppose $f\left(u, Du\right) \cdot Du = 0$, then by Theorem 5.1 in [MY], $\tilde{u}$ can be extended across the north as a regular solution on the whole S^m. The value of $\tilde{u}$ at the north pole will be denoted simply by $u\left(\infty\right)$. Note that both conditions $\operatorname{div}(a\left(u, Du\right)) = 0$ and $f\left(u, Du\right) \cdot Du = 0$ are naturally satisfied by the two examples below. A unique property of the bubbles is a uniform lower bound for their energy. That is, there is a $\mu > 0$ such that for any nontrivial bubble u,

$$\int_{R^m} |Du|^m dx \geq \mu.$$

It is well-known that for a weakly convergence sequence $\{u_n\} \subset W^{1,m}\left(\Omega, R^k\right)$, $u_n \to u$ strongly if and only if the energy converges: $E_m\left(u_n\right) \to E\left(u\right)$, see [El] for example. Our main result describes the convergence behaviors of certain weakly convergent sequences, and accounts for all the energy loss with a finite number of bubbles.

Theorem 1.1. *Suppose that (H-I) and (H-II) are satisfied. If $\{u_n\} \subset W^{1,m}\left(\Omega, R^k\right)$ is a sequence that satisfies (1.4) and for some $p > \frac{m}{m-1}$,*

$$h_n,\ \operatorname{div}\left(a\left(u_n, Du_n\right)\right) \to 0 \text{ in } W^{-1,p},\ \text{as } n \to \infty,$$

then there exist a solution $u \in C^1\left(\Omega, R^k\right)$ of (1.1), a finite number l of bubbles $\omega_i \in C^1\left(S^m, R^k\right)$, l sequences of points $\left\{a_n^i\right\} \subset \Omega$, l sequences of positive numbers $\left\{\lambda_n^i\right\}$, $1 \leq i \leq l$, and a subsequence of $\{u_n\}$, still denoted by $\{u_n\}$, such that

(1) $\lim_{n\to\infty} E_m\left(u_n\right) = E_m\left(u\right) + \sum_{i=1}^{l} E_m\left(\omega_i\right)$,

(2) For $i \neq j$, $\max\left\{\frac{\lambda_n^i}{\lambda_n^j}, \frac{\lambda_n^j}{\lambda_n^i}, \frac{|a_n^i - a_n^j|}{\lambda_n^i + \lambda_n^j}\right\} \to \infty$ *as $n \to \infty$,*

(3) $\left\|u_n - \sum_{i=1}^{l}\left(\omega_i\left(\frac{\cdot - a_n^i}{\lambda_n^i}\right) - \omega_i\left(\infty\right)\right)\right\|_{W^{1,m}} \to 0$ *as $n \to \infty$.*

We now apply this result to two examples: m-harmonic maps to a homogeneous space and m-harmonic maps with a constant volume.

Application to m-Harmonic Maps

Let Ω^m and N^q be two closed Riemannian manifolds. Assume N is homogeneous, and isometrically embedded into some R^k, $k \geq 2$. Denote by $W^{1,m}\left(\Omega, N\right)$ the set of all $v \in W^{1,m}\left(\Omega, R^k\right)$ with $v\left(x\right) \in N$ for a.e. $x \in \Omega$. Similarly, for a given $u \in W^{1,m}\left(\Omega, N\right)$, $W^{1,m}\left(\Omega, T_u N\right)$ denotes the space of all $v \in W^{1,m}\left(\Omega, R^k\right)$ with $v\left(x\right) \in T_{u(x)}N$ for a.e $x \in \Omega$. An m-harmonic map is a critical point of $E_m\left(u\right)$ in $W^{1,m}\left(\Omega, N\right)$, which satisfies

$$-mE_m'\left(u\right) \equiv \operatorname{div}\left(|Du|^{m-2}Du\right) + |Du|^{m-2}A\left(u\right)\left(Du, Du\right) = 0,$$

in $W^{-1,m'}(\Omega, T_uN) = \left[W_0^{1,m}(\Omega, T_uN)\right]'$, where $m' = \frac{m}{m-1}$, and A is second fundamental form of N in R^k.

A Palais-Smale sequence $\{u_n\} \subset W^{1,m}(\Omega, N)$ of E_m then satisfies

$$\limsup_{n\to\infty} E_m(u_n) < \infty$$

and

$$-\text{div}\left(|Du_n|^{m-2}Du\right) = |Du_n|^{m-2}A(u_n)(Du_n, Du_n) + h_n,$$

where $h_n \to 0$ in $W^{-1,m'}\left(\Omega, R^k\right)$.

Under a slightly stronger condition on h_n, Theorem 1.1 implies the following

Theorem 1.2. *Suppose that Ω is a closed manifold, N is a homogeneous manifold and $\{u_n\} \subset W^{1,m}(\Omega, N)$ is a Palais-Smale sequence such that for some $p > m'$,*

$$\sup_n \|h_n\|_{W^{-1,p}} < \infty.$$

Then there exist an m-harmonic map $u \in C^1(\Omega, N)$ and a finite number l of m-harmonic maps $\omega_i \in C^1(S^m, N)$, l sequences of points $\left\{a_n^i\right\} \subset \Omega$, l sequences $\left\{\lambda_n^i\right\}$ of positive numbers, $1 \le i \le l$, and a subsequence of $\{u_n\}$, still denoted as $\{u_n\}$, such that

(1) $\lim_{n\to\infty} E_m(u_n) = E_m(u) + \sum_{i=1}^l E_m(\omega_i)$,

(2) For $i \ne j$, $\max\left\{\frac{\lambda_n^i}{\lambda_n^j}, \frac{\lambda_n^j}{\lambda_n^i}, \frac{|a_n^i - a_n^j|}{\lambda_n^i + \lambda_n^j}\right\} \to \infty$ *as* $n \to \infty$,

(3) $\left\|u_n - \sum_{i=1}^l \left(\omega_i\left(\frac{\cdot - a_n^i}{\lambda_n^i}\right) - \omega_i(\infty)\right)\right\|_{W^{1,m}} \to 0$ *as* $n \to \infty$.

The condition that h_n is bounded in $W^{-1,p}$ for some $p > m'$ can not be droped, as shown by the example in [Pt].

In a pioneer work, Sacks-Uhlenbeck first developed in [SaU] the blow-up method to study a perturbed energy functional. As an application, they obtained the existence of minimal immersions into a Riemannian manifold. Struwe [Sm1] obtained a similar result for a class of solutions of harmonic map heat flows on surfaces. Jost [Jj] described the bubbling process of a mini-max scheme for maps from a surface to a closed manifold; see also the paper of Parker [Pt]. Bethuel [Bf] showed that the weak limits of Palais-Smale sequences of energy of maps on a surface are also harmonic maps, but he did not describe the bubbling process. Qing [Qj] described the bubbling behavior of certain Palais-Smale sequences of maps from a surface to standard spheres. Our result can be considered as a generalization of these results to higher dimension cases. In a forthcoming paper [Wc], the second author proves the same result as Theorem 1.2 for maps from a surface to a general compact manifold N.

As a corollary, we obtain the strong convergence of certain Palais-Smale sequences.

Corollary 1.3. *Suppose that Ω is a closed manifold, N is a homogeneous manifold and $\{u_n\} \subset W^{1,m}(\Omega, N)$ is a Palais-Smale sequence satisfying the conditions in Theorem 1.2. Then there exists a* *subsequence of $\{u_n\}$ strongly convergent to an m-harmonic map $u \in C^1(\Omega, N)$, provided any one of the following conditions holds:*

(1) If $\limsup_{n\to\infty} \int_\Omega |Du_n|^m \, dx < \int_\Omega |Du|^m + \mu$, where $\mu > 0$ is the largest lower bound of m-energy of non-constant m-harmonic maps from S^m to N.

(2) N supports a strictly convex function f, namely, there exists a number $c_0 > 0$ such that

$$\text{Hessian } (f) \geq c_0 h,$$

where h is the metric on N.

Application to m-Harmonic Maps with Constant Volumes

Let $\Omega \subseteq R^m$ be a smooth domain. An m-harmonic map $u \in W^{1,m}\left(\Omega, R^{m+1}\right)$ with constant volume is a critical point of $\int_\Omega |Du|^m$ subject to constant volume enclosed by the cone generated by $u(\Omega)$ with vertex $0 \in R^{m+1}$. The volume can be expressed as

$$V(u) = \frac{1}{m+1} \int_{R^m} u \cdot u_1 \wedge \cdots \wedge u_m,$$

where $u_1 \wedge \cdots \wedge u_m$ is the cross product of derivatives $u_i = \frac{\partial u}{\partial x^i}$.

An m-harmonic map with constant volume satisfies the equation

$$-\text{div}\left(|Du|^{m-2} Du\right) = H u_1 \wedge \cdots \wedge u_m, \tag{1.6}$$

for some constant H. See [MY] for details. It has been noticed that the right hand side

$$f(u, Du) = H u_1 \wedge \cdots \wedge u_m$$

can be written in the form (1.2). Specifically, we have

$$\begin{aligned} f^i &= H(-1)^{i+1} \frac{\partial\left(u^1, ..., \hat{u^i}, ..., u^{m+1}\right)}{\partial(x_1, ..., x_m)} \\ &= \sum_{l=1}^m a^{il} \frac{\partial u^{i+1}}{\partial x_l}, \end{aligned}$$

where $i = 1, ..., n+1$ ($u^{n+2} = u^1$), and

$$a^{il} = H(-1)^{i+l} \frac{\partial\left(u^1, ..., \hat{u^i}, \hat{u^{i+1}}, ..., u^{m+1}\right)}{\partial(x_1, ..., \hat{x}_l, ... x_m)}.$$

It is easy to check that

$$\text{div}\left(a^i\right) = \sum_{l=1}^m \frac{\partial a^{il}}{\partial x_l} = 0$$

for any $u \in W^{1,m}\left(R^m, R^{m+1}\right)$; see [Db]. In the case $\Omega = R^m$, $W^{1,m}\left(S^m, R^{m+1}\right)$ can be identified with $W^{1,m}\left(R^m, R^{m+1}\right)$ through the stereographic projection $\pi : R^m \to S^m$. Suppose u satisfies (1.6) on R^m, then and $u \circ \pi$ also satisfies (1.6) on $S^m \setminus \{\text{north pole}\}$. By the regularity results in [MY], both u and $u \circ \pi$ have to be regular ($C^{1,\alpha}$, $0 < \alpha < 1$) on R^m and on S^m respectively, and $u(\infty) = u \circ \pi(\text{north pole})$ esists. We state a result of application of Theorem 1.1 to the case $\Omega = S^m$:

Theorem 1.4. *Suppose $\{u^n\} \subset W^{1,m}\left(S^m, R^{m+1}\right)$ is a bounded sequence of solutions of*

$$-\operatorname{div}\left(|Du^n|^{m-2} Du\right) = Hu_1^n \wedge \cdots \wedge u_m^n + h^n, \tag{1.7}$$

with $h^n \to 0$ in $W^{-1,p}$, as $n \to \infty$, for some $p > \frac{m}{m-1}$. Then there exist a solution $u \in C^1\left(S^m, R^k\right)$ of (1.6), a finite number l of bubbles $\omega_i \in C^1\left(S^m, R^k\right)$, l sequences of points $\left\{a_n^i\right\} \subset S^m$, l sequences of positive numbers $\left\{\lambda_n^i\right\}$, $1 \le i \le l$, and a subsequence of $\{u_n\}$, still denoted by $\{u_n\}$, such that

(1) $\lim_{n\to\infty} E_m(u_n) = E_m(u) + \sum_{i=1}^l E_m(\omega_i)$

(2) For $i \neq j$, $\max\left\{\frac{\lambda_n^i}{\lambda_n^j}, \frac{\lambda_n^j}{\lambda_n^i}, \frac{|a_n^i - a_n^j|}{\lambda_n^i + \lambda_n^j}\right\} \to \infty$ as $n \to \infty$

(3) $\left\| u_n - \sum_{i=1}^l \left(\omega_i\left(\frac{\cdot - a_n^i}{\lambda_n^i}\right) - \omega_i(\infty)\right)\right\|_{W^{1,m}} \to 0$ as $n \to \infty$.

Remark. In the case $\partial\Omega \neq \emptyset$, bubbling phenomena also occur and can be described in the similar manner as in Theorem 1.1. In such a case, one has to include bubbles on the half space (or equivalently, on the unit ball B, through a conformal transformation), which are solutions of (1.5) on $R^+ \times R^{m-1}$ (or on B, resp.) and constant on $R^+ \times \{0\}$ (on ∂B, resp.). For some equations, there are no nontrivial bubbles on the half space (or the unit ball) and then conclusions of Theorem 1.1 hold for Ω with or without boundary. This is the case for (1.6) when $n = 2$, where Wente [Wh2] proved that the only solution of $-\Delta u = 2u_1 \wedge u_2$ in $W_0^{1,2}\left(D, R^3\right)$ is 0, where $D = \left\{x \in R^2 : |x| < 1\right\}$. Wente's this result was used in the paper of Brezis and Coron [BC], which established the bubbling phenomena of solution sequences $\{u^n\} \subset W_0^{1,2}\left(D, R^3\right)$ satisfying $\Delta u^n = 2u_1^n \wedge u_2^n + f^n$ with $f^n \to 0$ in $W^{-1,2}$. For harmonic maps on a disc with constant boundary values, Lemaire proved that they must be constant [Ll]. It would be interesting to know whether the results of Lemaire and Wente hold in higher dimensional cases. For example, suppose $u \in W^{1,m}\left(B, R^{m+1}\right)$ satisfies (1.6) (or (1.1)) and $u = \frac{\partial u}{\partial n} = 0$ on ∂B, where $B = \{x \in R^m : |x| < 1\}$. Is $u \equiv 0$? This is closely related to the unique continuation problem for m-Laplacian equations.

The idea of proof for all these Theorems is based on two steps: first, we prove the so-called ϵ-compactness lemma which is a consequence of the ϵ-continuity estimates; second, we study the problem about bubbles over bubbles and show that there is no energy concentrating in neck regions (to be specified below in Sections 2, 3, 4).

The paper is organized as follows. In §2, we prove the ϵ-continuity estimate. As a corollary, we prove the regularity of the solutions being considered. In §3, we obtain an ϵ-compactness lemma and some comparison lemmas that are necessary for the proof of Theorem 1.1. In §4, we prove the main results, by analyzing the concentration density of the energy, using various estimates we obtained in Sections 2 and 3.

Since we only consider the cases without boundary, all the needed estimates are of local version. For this reason and for simplicity, in Sections 2 and 3, we assume Ω is a smooth domain in R^m with standard metric. As for notations, constants are generically denoted by C or C_i; they may change from line to line. We will denote by $o(1)$ a quantity or a sequence that goes to 0 as the variable (index) goes to infinity. $B_r(x)$ denotes the ball in R^m centered at x and of radius r; $B_r = B_r(0)$.

§2 ϵ-Continuity Estimates

The hypothesis I on f is used to show that $f\left(u, Du\right)$ is in Hardy space $\mathcal{H}^1_{\text{loc}}$. For the definition and properties of $\mathcal{H}^1_{\text{loc}}\left(\Omega, R\right)$ and BMO(R^m, R), please see [Ss] or [CLMS] and the references there. Here we have

Proposition 2.1. *Suppose $u \in W^{1,m}\left(\Omega, R^k\right)$ such that*

$$\bar{h} \equiv div\left(a\left(u, Du\right)\right) \in W^{-1,p}$$

for some $p > m'$. Then $a\left(u, Du\right) \cdot Db\left(u\right) \in \mathcal{H}^1_{\text{loc}}\left(\Omega, R^k\right)$. Moreover, for any compact subset $K \subset\subset \Omega$, there is a constant C_K such that

$$\|a\left(u, Du\right) \cdot Db\left(u\right)\|_{\mathcal{H}^1(K)} \leq C_K \left[\|Du\|^m_{m,\Omega} + \left\|\bar{h}\right\|^{m'}_{W^{-1,p}}\right]. \tag{2.1}$$

Proof. This was essentially proved in [CLMS]. To trace the estimate, we sketch the proof. Take a $\phi \in C^\infty_0\left(R^m, R\right)$ such that $\int_{R^m} \phi = 1$ and $\text{spt}\left(\phi\right) \subset B\left(0, 1\right)$. For given $K \subset\subset \Omega$, $x \in K$ and $r < \text{d}\left(x, \partial\Omega\right)$, we define $\phi_r\left(y\right) = r^{-m}\phi\left(\frac{y-x}{r}\right)$. Then by integration of parts,

$$\left[a\left(u, Du\right) \cdot Db\left(u\right)\right] \star \phi\left(x\right) = \int_{R^m} \left[a\left(u, Du\right) \cdot Db\left(u\right)\right]\left(y\right) \phi_r\left(y\right) dy$$

$$= \int_{R^m} \left(b\left(u\right) - b\left(u\right)_{x,r}\right) \bar{h} \cdot \phi_r + \int_{R^m} \left(b\left(u\right) - b\left(u\right)_{x,r}\right) a\left(u, Du\right) \cdot D\phi_r = I + II.$$

Here $b\left(u\right)_{x,r} = ⨍_{B_r(x)} b\left(u\right)$. We estimate I and II separately.

$$\begin{aligned} |II| &\leq Cr^{-(m+1)} \int_{B_r(x)} |Du|^{m-1} |b\left(u\right) - b\left(u\right)_{x,r}| \\ &\leq CM\left(|Du|^{(m-1)q}\right)^{\frac{1}{q}}\left(x\right) M\left(|Du|^s\right)^{\frac{1}{s}}\left(x\right), \end{aligned}$$

where q and s are chosen so that $1 < q < m'$, $s = \frac{mq'}{m+q'}$ and $1 < s < m$. Here $M\left(f\right)$ is the local maximal function defined by

$$M\left(f\right)\left(x\right) = \sup\left\{\frac{1}{|B_r\left(x\right)|} \int_{B_r(x)} |f| : B_r\left(x\right) \subset \Omega\right\}.$$

By the isomorphism theorem of $\Delta : W^{1,p}_0 \to W^{-1,p}$, we find an $H \in W^{1,p}_0\left(\Omega, R^k\right)$ such that $\Delta H = \bar{h}$ with $\|H\|_{W^{1,p}} \leq C \left\|\bar{h}\right\|_{W^{-1,p}}$. It follows

$$
\begin{aligned}
I &= \int_{R^m} \left(b(u) - b(u)_{x,r}\right) \Delta H \phi_r \\
&= -\int_{R^m} DHDb(u)\,\phi_r - \int_{R^m} DH \left(b(u) - b(u)_{x,r}\right) D\phi_r.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
|I| &\leq Cr^{-m} \int_{B_r(x)} |DH||Du| + Cr^{-(m+1)} \int_{B_r(x)} |DH||b(u) - b(u)_{x,r}|. \\
&= III + IV.
\end{aligned}
$$

For IV, choose $1 < q < p$, $1 < s < m$ such that $s = \frac{mq'}{m+q'}$. Then

$$
IV \leq C\left[M^{\frac{1}{q}}\left(|DH|^q\right) M^s\left(|Du|^{\frac{1}{s}}\right)\right].
$$

As for III, take $1 < \alpha < m$ and $\beta = \alpha'$ such that $m' < \beta < p$. Then

$$
III \leq CM\left(|Du|^\alpha\right)^{\frac{1}{\alpha}} M\left(|DH|^\beta\right)^{\frac{1}{\beta}}.
$$

By the Hardy-Littlewood maximum theorem (see Stein [Se]), Hölder inequality, and the definition of $\|\cdot\|_{\mathcal{H}^1(K)}$, we have

$$
\begin{aligned}
&\|a(u, Du) \cdot Db(u)\|_{\mathcal{H}^1(K)} \\
&\leq C\int_\Omega |Du|^m + C\int_K |DH|^{m'} + C\int_K M\left(|DH|^\beta\right)^{m'/\beta} \\
&\leq C\|Du\|^m_{m,\Omega} + C_K\left(\int_K |DH|^p\right)^{m'/p} + C_K\left(\int_K M\left(|DH|^\beta\right)^{\frac{p}{\beta}}\right)^{m'/p} \\
&\leq C\|Du\|^m_{m,\Omega} + C_K\|h\|^{m'}_{W^{-1,p}}.
\end{aligned}
$$

■

It was proved in Toro-Wang [TW] that any m-harmonic map to a Riemannian homogeneous space has Hölder continuous gradient. The same result was shown in [MY] for the solutions u of (1.1) that satisfy (H-1) and $\operatorname{div}(a(u, Du)) = 0$. Now we prove an energy decay lemma, which will be used to show C^0 continuity of solutions u of (1.4).

Lemma 2.2. *Suppose $u \in W^{1,m}\left(B^m, R^k\right)$ is a solution of*

$$
-div\left(|Du|^{m-2}Du\right) = f(u, Du) + h, \quad \text{with } \bar{h} = \operatorname{div}(a(u, Du)), \tag{2.2}
$$

where f satisfies (H-1) and $\bar{h}, h \in W^{-1,p}$ for some $p > m'$. Then there exist positive numbers ϵ_0, $\theta_0 < 1$, $\alpha_0 < 1$ and C_0, independent of u, such that for every $x \in B^m_{\frac{1}{2}}$ and $r < \frac{1}{4}$, if $\int_{B^m} |Du|^m \leq \epsilon_0^m$, then

$$
\int_{B_r(x)} |Du|^m \leq \theta_0 \int_{B_{2r}(x)} |Du|^m + C_0\left[\|h\|^{m'}_{W^{-1,p}} + \left\|\bar{h}\right\|^{m'}_{W^{-1,p}}\right] r^{\alpha_0}. \tag{2.3}
$$

Proof. For given $x \in B_{\frac{1}{2}}$ and $r < \frac{1}{4}$, take $\eta \in C_0^1\left(B_{2r}\left(x\right), [0,1]\right)$ such that $\eta = 1$ in $B_r\left(x\right)$. Denote $A\left(x, r\right) = B_{2r}\left(x\right) \setminus B_r\left(x\right)$ and $u_{x,2r} = \fint_{A(x,r)} u$. Multiplying (2.2) by $\eta^m\left(u - u_{x,2r}\right)$ and integrating, we get that

(2.4)
$$\int_{B_1} |Du|^{m-2} Du \cdot D\left(\eta^m\left(u - u_{x,2r}\right)\right) = \int_{B_1} f\left(u, Du\right)\eta^m\left(u - u_{x,2r}\right) + \int_{B_1} h\eta^m\left(u - u_{x,2r}\right) = I + II.$$

Denote $E(u, 2r) = \int_{B_{2r}(x)} |Du|^m$. By Poincare inequality, Hölder inequality and the definition of BMO, we have

$$\left\|\eta^m\left(u - u_{x,2r}\right)\right\|_{W^{1,m}}^m \le CE(u, 2r);$$

$$\left\|\eta^{m-1}\left(u - u_{x,2r}\right)\right\|_{BMO} \le C\left(E(u, 2r)\right)^{\frac{1}{m}} \le C\epsilon_0;$$

$$\begin{aligned}
\left|\int_{R^m} \eta^m\left(u - u_{x,2r}\right)\right| &\le Cr^{m-1}\left(\int_{B_{2r}(x)} \left|\eta\left(u - u_{x,2r}\right)\right|^m\right)^{\frac{1}{m}} \\
&\le Cr^m\left(\int B_{2r}(x)\left|D\left(\eta\left(u - u_{x,2r}\right)\right)\right|^m\right)^{1/m} \\
&\le Cr^m\left(E(u, 2r)\right)^{\frac{1}{m}} \le C\epsilon_0.
\end{aligned}$$

It follows

$$\begin{aligned}
|II| &\le C\left\|h\right\|_{W^{-1,m'}(B_{2r}(x))}\left\|\eta^m\left(u - u_{x,2r}\right)\right\|_{W^{1,m}} \\
&\le C\left\|h\right\|_{W^{-1,m'}(B_{2r}(x))}\left(E(u, 2r)\right)^{\frac{1}{m}} \\
&\le C\epsilon_0\left\|h\right\|_{W^{-1,m'}(B_{2r}(x))}.
\end{aligned} \tag{2.5}$$

In order to estimate I, we define $\bar{f}\left(u, Du\right) = \left(E(u, 2r)\right)^{-1} f\left(u, Du\right)$ and pick up a $p_1 \in (m', p)$. Then Lemma 2.1 implies that $\bar{f}\left(u, Du\right) \in \mathcal{H}^1_{\text{loc}}\left(B_1\right)$ and for $K = \text{spt}\left(\eta\right)$ we have

$$\begin{aligned}
\left\|\bar{f}\left(u, Du\right)\right\|_{\mathcal{H}^1(K)} &\le C\left(E(u, 2r)\right)^{-1}\left[\int_{B_{2r}(x)} |Du|^m + \left\|\bar{h}\right\|_{W^{-1,p_1}(B_{2r}(x))}^{m'}\right] \\
&\le C + C\left(E(u, 2r)\right)^{-1}\left\|\bar{h}\right\|_{W^{-1,p_1}(B_{2r}(x))}^{m'}.
\end{aligned} \tag{2.6}$$

Then we have

$$\begin{aligned}
I &= E(u, 2r)\int_{B_1} \bar{f}\left(u, Du\right)\eta^m\left(u - u_{x,2r}\right) \\
&= E(u, 2r)\int_{R^m} \eta\left(\bar{f}\left(u, Du\right) - \mu\right)\eta^{m-1}\left(u - u_{x,2r}\right) \\
&\quad + \mu E(u, 2r)\int_{R^m} \eta^m\left(u - u_{x,2r}\right).
\end{aligned}$$

Here $\mu = \int \eta \bar{f}(u, Du) / \int \eta$. Use Proposition [S] in [TW] to conclude that $\eta \left(\bar{f}(u, Du) - \mu\right) \in \mathcal{H}^1(R^m)$ with

$$\begin{aligned}\left\|\eta\left(\bar{f}(u, Du) - \mu\right)\right\|_{\mathcal{H}^1(R^m)} &\le C\left(1 + \left\|\bar{f}(u, Du)\right\|_{\mathcal{H}^1(K)}\right)\\ &\le C\left[1 + (E(u, 2r))^{-1}\left\|\bar{h}\right\|^{m'}_{W^{-1,p_1}(B_{2r}(x))}\right].\end{aligned}$$

$$|\mu| \le Cr^{-m}\left\|\bar{f}(u, Du)\right\|_{\mathcal{H}^1(K)}.$$

By the duality of $\mathcal{H}^1$ and BMO, I can be estimated as follows.

$$\begin{aligned}|I| &\le CE(u, 2r)\left\|\eta\left(\bar{f}(u, Du) - \mu\right)\right\|_{\mathcal{H}^1}\left\|\eta^{m-1}(u - u_{x,2r})\right\|_{BMO}\\ &\quad + |\mu| E(u, 2r)\left|\int \eta^m (u - u_{x,2r})\right|\\ &\le C\epsilon E(u, 2r) + C\epsilon\left\|\bar{h}\right\|^{m'}_{W^{-1,p_1}(B_{2r}(x))}.\end{aligned} \tag{2.7}$$

For the left hand side of (2.4), we have

$$\begin{aligned}&\int_{B_1}\eta^m |Du|^m + m\int_{B_1}\eta^{m-1}D\eta|Du|^{m-2}Du\cdot(u - u_{x,2r})\\ &\ge \int_{B_r}|Du|^m - C\int_{A(x,r)}|Du|^m.\end{aligned}$$

Here we used Hölder and Poincare inequalities. Use Hölder inequality for the h and $\bar{h}$ terms in (2.5) and (2.7) and combine (2.4)-(2.7). We obtain

$$\begin{aligned}\int_{B_r(x)}|Du|^m &\le C\int_{A(x,r)}|Du|^m + C\epsilon_0\int_{B_{2r}(x)}|Du|^m\\ &\quad + C\epsilon_0\|h\|^{m'}_{W^{-1,p}(B_1)}r^{\alpha_0},\end{aligned}$$

$\alpha_0 = \min\left\{m\left(\frac{1}{m'} - \frac{1}{p}\right), m\left(\frac{m'}{p_1} - \frac{m'}{p}\right)\right\}$. Add $C\int_{B_r(x)}|Du|^m$ to the above inequality. We have

$$\int_{B_r(x)}|Du|^m \le \theta_0\int_{B_{2r}(x)}|Du|^m + C\epsilon_0\left\|\bar{h}, h\right\|_{W^{-1,p}(B_1)}r^{\alpha_0},$$

where $\theta_0 = \frac{C + C\epsilon_0}{C+1} < 1$, if we choose $\epsilon_0 < \frac{1}{2C}$. Hence (2.3) holds. ■

Theorem 2.3. *There exist positive constants ϵ_0, $\delta_0 < 1$ and C_1 such that if $u \in W^{1,m}\left(B^m, R^k\right)$ satisfies the conditions in Lemma 2.2, and $\int_{B_1}|Du|^m \le \epsilon_0^m$, then $u \in C^{\delta_0}\left(B_{1/2}, R^k\right)$ and $\|u\|_{C^{\delta_0}\left(B_{1/2}\right)} \le C_1$.*

Proof. For every $x \in B_{\frac{1}{2}}$ and $r < \frac{1}{4}$, if we define $F(x, r) = \int_{B_r(x)}|Du|^m$ then Lemma 2.2 implies that

$$F(x, r) \le \theta_0 F(x, 2r) + Cr^{\alpha_0},$$

where C depends on $p, p_1, \epsilon_0, \left\|\bar{h}, h\right\|_{W^{-1,p}(B_1)}$. By Lemma 8.23 in Gilbarg-Trudinger [GT], there are numbers $\beta_0 \in (0, 1)$ and $R \le \frac{1}{4}$, such that for all $r \le R$,

$$F(x, r) \le C\left(\frac{r}{r}\right)^{\beta_0}F(x, R) + \left(\frac{r}{R}\right)^{\alpha_0}.$$

By the Morrey's decay lemma (see Morrey [Mc]), $u \in C^{\delta_0}\left(B_{1/2}\right)$ for $\delta_0 = \min\{\alpha_0, \beta_0\}$ and $\|u\|_{C^{\delta_0}\left(B_{1/2}\right)} \le C_1$. ■

Corollary 2.4. *Suppose $u \in W^{1,m}\left(\Omega, R^k\right)$ is a solution of*

$$-div\left(|Du|^{m-2}Du\right) = f\left(u, Du\right) + h, \quad \text{with } \bar{h} = \operatorname{div}\left(a\left(u, Du\right)\right), \tag{2.2}$$

where f satisfies (H-1) and $\bar{h}, h \in W^{-1,p}$ for some $p > m'$. Then there exists a positive number δ_0, such that for every $u \in C^{1,\delta_0}(\Omega, R^k)$.

Proof. For every point in Ω, say $0 \in \Omega$, we take a small number $r > 0$ such that $\int_{B_r} |Du|^m < \epsilon_0^m$ as in Theorem 2.3. Consider $u_r \in W^{1,m}(B_1, R^k)$ defined by $u_r(x) = u(rx)$. Then u_r satisfies (2.2) with f being replaced by $f(u_r, Du_r)$ and h by $r^m h$ with $\int_{B_1} |Du_r|^m < \epsilon_0^m$. Therefore, Theorem 2.3 implies that $u_r \in C^{\delta_0}(B_{1/2})$ for some δ_0. So $u \in C^{\delta_0}(B_{r/2})$. C^{1,δ_0} regularity can be obtained by considering the equation satisfied by Du, as explained in [MY]. ■

We end this section with a sufficient condition for strong convergence of Palais-Smale sequences.

Proposition 2.5. *Let $\{u_n\} \subset W^{1,m}\left(\Omega, R^k\right)$ be a sequence satisfying (1.4). If $u_n \to u$ in $L^\infty_{\text{loc}}\left(\Omega, R^k\right)$, then $u_n \to u$ in $W^{1,m}_{\text{loc}}\left(\Omega, R^k\right)$. In particular, u is a solution of (1.1).*

Proof. It suffices to prove that $\{u_n\}$ is a Cauchy sequence in $W^{1,m}_{\text{loc}}$. In order to do that, let ξ be any cut-off function in Ω. Multiplying both equations of u_n and u_l by $\xi^2\left(u_n - u_l\right)$ and subtracting one from the other, we get

$$\begin{aligned}\int_\Omega \left(|Du_n|^{m-2}Du_n - |Du_l|^{m-2}Du_l\right) D\left(\xi^2\left(u_n - u_l\right)\right) &= \int_\Omega \left(h_n - h_l\right)\xi^2\left(u_n - u_l\right) \\ &+ \int_\Omega \left(f\left(u_n, Du_n\right) - f\left(u_l, Du_l\right)\right)\xi^2\left(u_n - u_l\right).\end{aligned} \tag{2.8}$$

It is easy to see that the right hand side satisfies

$$\begin{aligned}|\text{RHS}| &\le \|h_n - h_l\|_{W^{-1,m'}} \left\|\xi^2\left(u_n - u_l\right)\right\|_{W^{1,m}} \\ &+ \left(\|Du_n\|_m^m + \|Du_l\|_m^m\right)\left\|\xi^2\left(u_n - u_l\right)\right\|_{L^\infty}. \\ &\to 0, \text{as } n, l \to \infty.\end{aligned}$$

Therefore, by the convexity of integrand and the strong convergence of u_n in L^m (see [HLM], for example), the left hand side of (2.8) $\ge \int_{B_r} |Du_n - Du_l|^m + o\left(1\right)$ (as $l, n \to \infty$). It follows

$$\int_{B_r} |Du_n - Du_l|^m \to 0, \text{as } l, n \to \infty.$$

■

§3 ϵ-Compactness Lemmas

This section consists of some preparatory lemmas for the proof of the main Theorem 1.1.

Lemma 3.1. *Suppose the hypotheses (H-I) and (H-II) hold. Then there is an* $\epsilon_0 > 0$ *such that if* $\{u_n\} \subset W^{1,m}\left(B^m, R^k\right)$ *satisfies*

$$-div\left(|Du_n|^{m-2}Du_n\right) = f\left(u_n, Du_n\right) + h_n, \tag{3.1}$$

with both $\bar{h}_n \equiv \operatorname{div}\left(a\left(u_n, Du_n\right)\right)$, $\|h_n\|_{W^{-1,m'}} \to 0$, $\sup\left\|\bar{h}_n, h_n\right\|_{W^{-1,p}} < \infty$ *for some* $p > m'$ *and* $\int_{B_1}|Du_n|^m \leq \epsilon_0^m$, *then* u_n *contains a subsequence that converges to* u *in* $W^{1,m}_{\text{loc}}\left(B_1, R^k\right)$. $u \in C^1\left(B_1, R^k\right)$ *and satisfies (1.1).*

Proof. By passing to a subsequence, we may assume that $u_n \to u$ weakly in $W^{1,m}$ and strongly in L^m. By Theorem 2.3, for any $0 < r < 1$, $\|u_n\|_{C^{\delta_0}(B_r)} \leq C_1$ for some δ_0 and C_1 independent of i. Therefore, by Arzela-Ascolli's compactness theorem, we can extract a subsequence of $\{u_n\}$ (denoted by $\{u_n\}$) such that $u_n \to u$ in $C^0\left(B_r, R^k\right)$. Now apply Proposition 2.5 to get the conclusion. ■

We will compare the energy of the solutions to (1.4) with the energy of m-harmonic functions. First, we calculate the energy of the m harmonic function on an annulus with constant boundary values.

For $0 < R_1 < R_2$, let $A\left(R_1, R_2\right) = \{x \in R^m : R_1 \leq |x| \leq R_2\}$ and $\partial A\left(R_1, R_2\right) = \partial B_{R_1} \cup \partial B_{R_2}$. We have

Lemma 3.2. *Suppose that* $u \in W^{1,m}\left(A\left(R_1, R_2\right), R^k\right)$ *satisfies*

$$\begin{aligned} div\left(|Dv|^{m-2}Dv\right) &= 0, \text{ in } A\left(R_1, R_2\right), \\ u|_{\partial B_{R_1}} &= a, \\ u|_{\partial B_{R_2}} &= b. \end{aligned} \tag{3.2}$$

Here a *and* b *are constants in* R^k. *Then*

$$\int_{A(R_1,R_2)} |Dv|^m = \frac{|a-b|^m}{\left(\log\frac{R_2}{R_1}\right)^{m-1}}. \tag{3.3}$$

Proof. From the convexity of $\int_{A(R_1,R_2)}|Dv|^m$, we know that the solution of (3.2) is unique. Since the values of v on boundary are constant, we consider the radial solution $v\left(x\right) = v\left(|x|\right)$. Then the equation of v becomes

$$\begin{aligned} v''\left(r\right) + \frac{1}{r}v'\left(r\right) &= 0, \\ v\left(R_1\right) &= a, \\ v\left(R_2\right) &= b. \end{aligned} \tag{3.4}$$

It follows $v\left(x\right) = C + \frac{b-a}{\log\frac{R_2}{R_1}}\log|x|$ for some C and

$$\begin{aligned} \int_{A(R_1,R_2)} |Dv|^m &= \int_{R_1}^{R_2} |v'\left(r\right)|^m r^{m-1}\,dr \\ &= \left(\frac{|a-b|}{\log\frac{R_2}{R_1}}\right)^m \log\frac{R_2}{R_1}. \end{aligned}$$

$\blacksquare$

Lemma 3.3. *Suppose f satisfies hypotheses (H-I) and (H-II) and $u \in W^{1,m}\left(\Omega, R^k\right)$ satisfies*

$$-div\left(|Du|^{m-2}Du\right) = f\left(u, Du\right) + h$$

with $\bar{h} \equiv \operatorname{div}\left(a\left(u, Du\right)\right)$, $h \in W^{-1,p}$ for some $p > \frac{m}{m-1}$. There exist $\epsilon_0 > 0$ and C such that if $\int_\Omega |Du|^m \leq \epsilon_0^m$, and for $\Omega_1 \subset\subset \Omega$, define

$$-div\left(|Dv|^{m-2}Dv\right) = 0, in\ \Omega_1;$$
$$v|_{\partial\Omega_1} = u|_{\partial\Omega_1}.$$

Then

$$\|Du\|^m_{m,\Omega_1} \leq C\,\|Dv\|^m_{m,\Omega_1} + C\,\|h\|^{m'}_{W^{-1,p}}\,, \tag{3.5}$$

Proof. Multiplying both equations of u and v by $u - v$ and subtracting them, we get that

$$\int_{\Omega_1}\left(|Du|^{m-2}Du - |Dv|^{m-2}Dv\right)D\left(u-v\right) = \int_{\Omega_1} h\left(u-v\right) + \int_{\Omega_1} f\left(u, Du\right)\left(u-v\right) = I + II. \tag{3.6}$$

With the identification of h with ΔH for $H \in W_0^{1,p}$, by Hölder inequality and Sobolev embedding theorem, we have

$$\begin{aligned}|I| &\leq C\int_{\Omega_1}|DH||D\left(u-v\right)| \\ &\leq C\,\|DH\|_{m',\Omega_1}\|D\left(u-v\right)\|_{m,\Omega_1} \\ &\leq C\,\|h\|_{W^{-1,m'}(\Omega_1)}\|D\left(u-v\right)\|_{m,\Omega_1}\,.\end{aligned}$$

Let $\xi \in C_0^\infty\left(\Omega, R\right)$ be such that $\xi \equiv 1$ on Ω_1. Then again by Proposition [S] in [TW], we have

$$\begin{aligned}|II| &= |\int_{R^m} f\left(u, Du\right)\xi\left(u-v\right)| \\ &\leq C\,\|f\left(u, Du\right) - \mu\|_{\mathcal{H}^1(R^m)}\|u-v\|_{\mathrm{BMO}} + \mu\int_{R^m}\xi|u-v| \\ &\leq C\left(\|Du\|^m_{m,\Omega_1} + \left\|\bar{h}\right\|^{m'}_{W^{-1,p}}\right)\|D\left(u-v\right)\|_{m,\Omega_1}\,.\end{aligned}$$

Here $\mu = \left(\int\xi\right)^{-1}\int\xi f\left(u, Du\right)$, and we used Sobolev inequality and the fact that $|\mu|$ is less than the Hardy norm of $f\left(u, Du\right)$. Therefore (3.6) implies

$$\|D\left(u-v\right)\|^m_{m,\Omega_1} \leq C\left(\|Du\|^m_{m,\Omega_1} + \left\|\bar{h}, h\right\|^{m'}_{W^{-1,p}}\right)\|D\left(u-v\right)\|_{m,\Omega_1}\,.$$

i.e.,

$$\|D\left(u-v\right)\|^{m-1}_{m,\Omega_1} \leq C\left(\|Du\|^m_{m,\Omega_1} + \left\|\bar{h}, h\right\|^{m'}_{W^{-1,p}}\right). \tag{3.7}$$

On the other hand,

$$\|Du\|_{m,\Omega_1}^{m-1} \leq 2^{m-1}\left(\|D(u-v)\|_{m,\Omega_1}^{m-1} + \|Dv\|_{m,\Omega_1}^{m-1}\right)$$

Using (3.7), we get

$$\|Du\|_{m,\Omega_1}^{m-1} \leq C\|Du\|_{m,\Omega_1}^{m} + C\|Dv\|_{m,\Omega_1}^{m-1} + C\left\|\bar{h}, h\right\|_{W^{-1,p}(\Omega_1)}^{m'}.$$

Choose ϵ_0 small enough, then (3.5) follows. ■

Finally, we give the proof that the energy of bubbles has a lower bound.

Proposition 3.4. *Suppose that (H-I) and (H-II) are satisfied.*

$$\mu = \inf\left\{\int_{S^m} |Du|^m : u : S^m \to R^k \text{ is non-constant solution of (1.5)}\right\} > 0.$$

Proof. Suppose $\mu = 0$, then by its definition, there exists a sequence of non-constant solutions $\{u_n\}$: $S^m \to R^k$ such that $\int_{S^m} |Du_n|^m \to 0$. By the regularity results in [MY][TW], $\{u_n\}$ are $C^{1,\alpha}$ with $\|u_n\|_{C^{1,\alpha}} \leq C$ with $0 < \alpha < 1$, C independent of i. Since u_n are not constant, there exist $\{p_n\} \subset S^m$ such that $|Du_n|(p_n) \neq 0$. Using the conformal invariance of m-energy, we can assume, by composing u_i with suitable conformal transformation of S^m, that $|Du_n|(p) = 1$ for some fixed $p \in S^m$. By passing to a subsequence, we can assume $u_n \to u$ in $C^1 \cap W^{1,m}$. This implies that u =constant and $|Du|(p) = 1$, a contradiction. ■

§4 **Proofs of Main Theorems**.

Proof of Theorem 1.1. Without loss of generality, we assume that there exists $u \in W^{1,m}\left(\Omega, R^k\right)$ such that $u_n \to u$ weakly in $W^{1,m}$ and strongly in L^m. For clarity, we divide the proof into four steps.

Step 1. Define the concentration set $\Sigma \subset \Omega$ by

$$\Sigma = \cap_{r>0}\left\{x \in \Omega : \liminf_{n\to\infty}\int_{B_r(x)} |Du_n|^m > \epsilon_0^m\right\},$$

where ϵ_0 is the same number as in Theorem 2.3. It follows from a standard argument that Σ is a finite subset of Ω; see [SaU] or [TW]. Furthermore, for any $x_0 \notin \Sigma$, there exists $r_0 > 0$ such that

$$\liminf_{n\to\infty}\int_{B_{r_0}(x_0)} |Du_n|^m \leq \epsilon_0^m.$$

Thus the ϵ-compactness Lemma 3.1 implies that a subsequence of $\{u_n\}$, still denoted by $\{u_n\}$, satisfies

$$u_n \to u \text{ in } W_{\text{loc}}^{1,m}\left(B_{r_0}(x_0)\right) \cap C_{\text{loc}}^0\left(B_{r_0}(x_0)\right).$$

It follows

$$u_n \to u \text{ in } W_{\text{loc}}^{1,m} \cap C_{\text{loc}}^0(\Omega \setminus \Sigma).$$

In particular, u satisfies the equation (1.1) on $\Omega \setminus \Sigma$ and so on Ω (isolated singularities are removable). By the regularity results in [TW][MY], u is $C^{1,\alpha}$ on Ω.

Step 2. As in Brezis-Coron [BC], we define, for $0 < \delta_1 < \frac{1}{2} \min \{d(x_1, x_i) : 2 \le i \le l\}$,

$$Q_n(t) = \sup_{x \in B_{\delta_1}(x_1)} \int_{x + tB_1} |Du_n|^m. \tag{4.1}$$

It is easy to see that there exist sequences $\{a_n^1\} (\subset B_{\delta_1}(x_1)) \to x_1$ and $\lambda_n^1 \to 0$ such that

$$Q_n(\lambda_n^1) = \int_{a_n^1 + \lambda_n^1 B_1} |Du_n|^m = \frac{\epsilon_0^m}{2}. \tag{4.2}$$

Define the rescaling functions by

$$v_n : (\lambda_n^1)^{-1} (\Omega \setminus \{a_n^1\}) \to R^k \text{ by } v_n(x) = u_n(a_n^1 + \lambda_n^1 x),$$

then by conformal invariance of the energy,

$$\int_{(\lambda_n^1)^{-1}(\Omega \setminus \{a_n^1\})} |Dv_n|^m = \int_\Omega |Du_n|^m. \tag{4.3}$$

$$\int_{B_1(x)} |Dv_n|^m \le \frac{\epsilon_0^m}{2}, \tag{4.4}$$

for all $x \in R^m$ and with equality when $x = 0$. Moreover,

$$-\mathrm{div}\left(|Dv_n|^{m-2} Dv_n\right) = f(v_n, Dv_n) + (\lambda_n^1)^m h_n. \tag{4.5}$$

From (4.3) to (4.5), Lemma 3.1 applies to v_n. We have for some $\omega_1 \in W^{1,m}(R^m, R^k)$,

$$v_n \to \omega_1 \text{ in } C^0_{\mathrm{loc}} \cap W^{1,m}_{\mathrm{loc}}(R^m, R^k). \tag{4.6}$$

In particular, $\int_{B_1} |D\omega_1|^m = \frac{\epsilon_0^m}{2}$. So ω_1 is a nontrivial bubble. Repeating this process to $x_2, ..., x_l$, we get bubble solutions $\omega_i : S^m \to R^k$ and sequences $\{a_n^i\} \to x_i$, and $\lambda_n^i \to 0$, as $n \to \infty$, such that

$$u_n(a_n^i + \lambda_n^i \cdot) \to \omega_i \text{ in } C^0_{\mathrm{loc}} \cap W^{1,m}_{\mathrm{loc}}(R^m, R^k) \ . \tag{4.7}$$

Step 3: Define $w_n = u_n - \sum_{i=1}^l \left[\omega_i\left(\frac{\cdot - a_n^i}{\lambda_n^i}\right) - \omega_i(\infty)\right]$, where ∞ denotes north pole in S^m and $\frac{\cdot - a_n^i}{\lambda_n^i} = (\lambda_n^i)^{-1} \exp_{a_n^i}^{-1}(\cdot)$. We claim that

$$\int_\Omega |Dw_n|^m = \int_\Omega |Du_n|^n - \sum_{i=1}^l \int_{R^m} |D\omega_i|^m + o(1). \tag{4.8}$$

In fact, for $0 < \delta < \frac{1}{2}\min\{d(x_i, x_j) : i \neq j\}$ we have

$$\int_{B_\delta(x_i)} |Dw_n|^m = \int_{B_{R_i\lambda_n^i}(a_n^i)} |Dw_n|^m + \int_{B_\delta(x_i)\setminus B_{R_i\lambda_n^i}(a_n^i)} |Dw_n|^m \\ = I + II. \tag{4.9}$$

1/m Note that ω_i and ω_j have disjoint supports for $j \neq i$. It follows by a change of variables,

$$\int_{B_{R_i\lambda_n^i}(a_n^i)} |D\omega_j|^m = o(1). \tag{4.10}$$

Hence

$$I = \int_{B_{R_i\lambda_n^i}(a_n^i)} |Dw_n|^m \\ = \int_{B_{R_i\lambda_n^i}(a_n^i)} \left|D\left(u_n - \omega_i\left(\frac{\cdot - a_n^i}{\lambda_n^i}\right)\right)\right|^m + o(1) \\ = \int_{B_{R_i}} |D\left(u_n\left(a_n^i + \lambda_n^i\cdot\right) - \omega_i\right)|^m. \tag{4.11}$$

On the other hand, by local strong convergence of $u_n\left(a_n^i + \lambda_n^i\cdot\right)$ to ω_i in $W^{1,m}$, we have the following identity (see page 11 in Evans [El]) provided that R_i is chosen to be sufficiently large.

$$\int_{B_{R_i}} |D\left(w_n\left(a_n^i + \lambda_n^i\cdot\right) - \omega_i\right)|^m = \int_{B_{R_i}} |Du_n\left(a_n^i + \lambda_n^i\cdot\right)|^m - \int_{B_{R_i}} |D\omega_i|^m + o(1). \tag{4.12}$$

$$= \int_{B_{R_i\lambda_n^i}(a_n^i)} |Du_n|^m - \int_{R^m} |D\omega_i|^m + o(1). \tag{4.9}$$

It follows from (4.9)-(4.12) that

$$I = \int_{B_{R_i\lambda_n^i}(a_n^i)} |Du_n|^m - \int_{R^m} |D\omega_i|^m + o(1). \tag{4.10}$$

On the other hand, for $j \neq i$

$$\int_{B_\delta(x_i)\setminus B_{R_i\lambda_n^i}(a_n^i)} \left|D\omega_j\left(\frac{\cdot - a_n^j}{\lambda_n^j}\right)\right|^n \leq \int_{B_\delta(x_i)} \left|D\omega_j\left(\frac{\cdot - a_n^j}{\lambda_n^j}\right)\right|^m \tag{4.14}$$

$$\leq \int_{R^m\setminus B_{\delta\lambda_n^j}} |D\omega_j|^m = o(1).$$

We choose R_i sufficently large so that

$$\int_{R^m\setminus B_{R_i}} |D\omega_i|^m = o(1) \tag{4.15}$$

(4.14) and (4.15) imply that

$$|II| \leq \int_{B_\delta(x_i)\setminus B_{R_i\lambda_n^i}(a_n^i)} |Du_n|^m + o(1)\,. \tag{4.16}$$

Combining (4.10) with (4.16), we get, for $1 \leq i \leq l$, that

$$\int_{B_\delta(x_i)} |Dw_n|^m = \int_{B_\delta(x_i)} |Du_n|^m - \int_{R^m} |D\omega_i|^m + o(1)\,. \tag{4.17}$$

On $\Omega \setminus \cup_{i=1}^l B_\delta(x_i)$, we have $u_n \to u$ in $W^{1,m}$. Which implies that

$$\int_{\Omega\setminus\cup_{i=1}^l B_\delta(x_i)} |Dw_n|^m = \int_{\Omega\setminus\cup_{i=1}^l B_\delta(x_i)} |Du_n|^m + o(1)\,. \tag{4.18}$$

So, the combination of (4.17) and (4.18) implies the claim (4.8).

Step 4

Case 1: If $\lim_{n\to\infty}\int_\Omega |Dw_n|^m = \int_\Omega |Du|^m$, then we already conclude that $w_n \to u$ in $W^{1,m}(\Omega, R^k)$. Otherwise, we have

Case 2: $\lim_{n\to\infty}\int_\Omega |Dw_n|^m > \int_\Omega |Du|^m$. Let $\Sigma_1 \subset \Omega$ be the set of concentration of w_n. In fact, from (4.18) we know that $\Sigma_1 \subset \Sigma$. We assume that $\Sigma_1 = \{x_1, \cdots, x_{l'}\}$ for some $1 \leq l' \leq l$. Pick $x_1 \in \Sigma_1$, then we have

$$\limsup_{t\to 0}\lim_{n\to\infty}\int_{B_t(x_1)} |Dw_n|^m = \epsilon_1^m > 0. \tag{4.19}$$

1/m Define Q_n for w_n, then there exist $\{a_n^{l+1}\} \subset B_\delta(x_1)$ and $\{\lambda_n^{l+1}\} \subset R$ such that

$$Q_n\left(\lambda_n^{l+1}\right) = \int_{a_n^{l+1}+\lambda_n^{l+1}B^m} |Dw_n|^m = \min\left\{\frac{\epsilon_1^m}{2}, \frac{\epsilon_0^m}{2}\right\}. \tag{4.20}$$

It is easy to see that $\lambda_n^{l+1} \geq \lambda_n^1$ and $\lambda_n^{l+1} \to 0$ (otherwise, there exists $\lambda_0 > 0$ such that

$$\int_{x_1+\lambda_0 B^m} |Dw_n|^m \leq \frac{\epsilon_1^m}{2},$$

which contradicts with (4.19); moreover, $a_n^{l+1} \to x_1$ (otherwise, there exists concentrate point outside Σ_1). We define $w_n' : \left(\lambda_n^{l+1}\right)^{-1}\left(\Omega \setminus a_n^{l+1}\right) \to R^k$ by $w_n'(x) = w_n\left(a_n^{l+1} + \lambda_n^{l+1}x\right)$, as in Step 2. Then

$$\sup_n \int_{\left(\lambda_n^{l+1}\right)^{-1}\left(\Omega\setminus a_n^{l+1}\right)} |Dw_n'|^m = \sup_n \int_\Omega |Dw_n|^m < \infty. \tag{4.21}$$

$$\int_{B_1(x)} |Dw_n'|^m \leq \min\left\{\frac{\epsilon_1^m}{2}, \frac{\epsilon_0^m}{2}\right\}, \forall x \in R^m;\ \text{with equality for } x = 0. \tag{4.22}$$

We may assume that $w_n' \to \omega_{l+1}$ weakly in $W^{1,m}_{\text{loc}}\left(R^m, R^k\right)$. We now prove the following

Claim (1): For $1 \le i \le l$,

$$\max\left\{\frac{\lambda_n^{l+1}}{\lambda_n^i}, \frac{\lambda_n^i}{\lambda_n^{l+1}}, \frac{|a_n^i - a_n^{l+1}|}{\lambda_n^i + \lambda_n^{l+1}}\right\} \to \infty. \tag{4.23}$$

(2): $w_n' \to \omega_{l+1}$ in $W^{1,m}_{\mathrm{loc}}\left(R^m, R^k\right)$ and ω_{l+1} is a non-trivial bubble.

Proof of the Claim: It suffices to prove (4.23) for $i = 1$. Suppose it did not hold, there would exist $R < \infty$ such that $R^{-1} \le \frac{\lambda_n^{l+1}}{\lambda_n^1} \le R$ and $\frac{|a_n^1 - a_n^{l+1}|}{\lambda_n^1 + \lambda_n^{l+1}} \le R$. Then

$$\begin{aligned}\int_{a_n^{l+1}+\lambda_n^{l+1}B_1} |D\omega_n|^m &\le \int_{a_n^1+(R^2+2R)\lambda_n^1 B_1} |D\omega_n|^m \\ &= \int_{B_{R^2+2R}} \left|D\left(u_n\left(a_n^1 + \lambda_n^1\cdot\right) - \omega_1\right)\right|^m \\ &\to 0,\end{aligned}$$

which contradicts to (4.20). So (4.23) holds. To prove the Claim (2), we consider the two cases of (4.23).

Case (a): There exists $M > 0$ such that

$$\text{either } M^{-1} \le \frac{\lambda_n^{l+1}}{\lambda_n^1} \le M, \quad \frac{|a_n^1 - a_n^{l+1}|}{\lambda_n^1 + \lambda_n^{l+1}} \to \infty, \quad \text{or } \frac{\lambda_n^{l+1}}{\lambda_n^1} \to \infty, \frac{|a_n^1 - a_n^{l+1}|}{\lambda_n^1 + \lambda_n^{l+1}} \to \infty. \tag{4.24}$$

In this case, $\left(a_n^{l+1} + \lambda_n^{l+1} B_R\right) \cap \left(a_n^1 + \lambda_n^1 B_R\right) = \emptyset$ for large $R > 0$. Therefore,

$$\int_{B_R} \left|D\omega_1\left(\frac{a_n^{l+1} - a_n^1 + \lambda_n^{l+1}\cdot}{\lambda_n^1}\right)\right|^m \le \int_{R^m \setminus B_R} |D\omega_1|^m \le o(1).$$

and

$$\int_{B_1(x)} \left|Du_n\left(a_n^{l+1} + \lambda_n^{l+1}\cdot\right)\right|^m \le \min\left\{\frac{\epsilon_1^m}{2}, \frac{\epsilon_0^m}{2}\right\} + o(1), \forall x \in R^m, \text{ with equality for } x = 0.$$

So we apply lemma 3.1 to conclude that $u_n\left(a_n^{l+1} + \lambda_n^{l+1}\cdot\right) \to \omega_{l+1}$ in $W^{1,m}_{\mathrm{loc}}\left(R^m, R^k\right)$ for some ω_{l+1} : $R^m \to R^k$, which is a bubble.

Case (b): There exists $M < \infty$ such that $\frac{\lambda_n^{l+1}}{\lambda_n^1} \to \infty$, $\frac{|a_n^{l+1} - a_n^1|}{\lambda_n^{l+1}} \le M$. For the simplicity, we assume that $a_n^{l+1} = a_n^1 = x_1$. Then, for all $\alpha > 0$ and $1 \le i \le l$

$$\int_{R^m \setminus B_\alpha} \left|D\omega_i\left(\frac{a_n^{l+1} - a_n^1 + \lambda_n^{l+1}\cdot}{\lambda_n^1}\right)\right|^m \le \int_{R^m \setminus B_{\alpha\lambda_n^{l+1}/\lambda_n^i}\left(\frac{a_n^{l+1}-a_n^1}{\lambda_n^1}\right)} |D\omega_i|^m = o(1), \tag{4.25}$$

which implies

$$\int_{B_1(x) \setminus B_\alpha} \left|Du_n\left(a_n^{l+1} + \lambda_n^{l+1}\cdot\right)\right|^m \le \min\left\{\frac{\epsilon_1^m}{2}, \frac{\epsilon_0^m}{2}\right\} + o(1), \forall x \in R^m,$$

with equality at $x = 0$. Again, by Lemma 3.1, we get $u_n\left(a_n^{l+1} + \lambda_n^{l+1}\cdot\right) \to \omega_{l+1}$ in $W^{1,m}_{\mathrm{loc}}\left(R^m \setminus B_\alpha\right)$ and ω_{l+1} is a non-constant solution, which extends to R^m by letting $\alpha \to \infty$. Moreover, for any $R > 0$,

$$\int_{B_R} \left|D\left(w_n\left(a_n^{l+1} + \lambda_n^{l+1}\cdot\right) - \omega_{l+1}\right)\right|^m = \int_{B_\alpha} \left|D\left(w_n\left(a_n^{l+1} + \lambda_n^{l+1}\cdot\right) - \omega_{l+1}\right)\right|^m + o(1) \tag{4.26}$$

$$= \int_{B_{\alpha\lambda_n^{l+1}}\left(a_n^{l+1}\right)} \left| D\left(w_n - \omega_{l+1}\left(\frac{\cdot - a_n^{l+1}}{\lambda_n^{l+1}}\right)\right)\right|^m + o\left(1\right).$$

On the other hand, for $\beta > 0$

$$\int_{B_{\beta\lambda_n^1}\left(a_n^1\right)} |Dw_n|^m = o\left(1\right), \tag{4.27}$$

$$\int_{B_{\beta\lambda_n^1}\left(a_n^1\right)} \left|D\omega_{l+1}\left(\frac{\cdot - a_n^{l+1}}{\lambda_n^{l+1}}\right)\right|^m = \int_{B_{\beta\lambda_n^1/\lambda_n^{l+1}}} |D\omega_{l+1}|^m = o\left(1\right). \tag{4.28}$$

Therefore, if we denote $A\left(a_n^1, \beta\lambda_n^1, \alpha\lambda_n^{l+1}\right) = B_{\alpha\lambda_n^{l+1}}\left(a_n^1\right) \setminus B_{\beta\lambda_n^1}\left(a_n^1\right)$, then

$$\int_{B_R} |D\left(w_n\left(a_n^{l+1} + \lambda_n^{l+1}\cdot\right) - \omega_{l+1}\right)|^m = \int_{A\left(a_n^1, \beta\lambda_n^1, \alpha\lambda_n^{l+1}\right)} \left|D\left(w_n - \omega_{l+1}\left(\frac{\cdot - a_n^{l+1}}{\lambda_n^{l+1}}\right)\right)\right|^m + o\left(1\right). \tag{4.29}$$

We choose α so small and β so large that

$$\int_{B_\alpha} |D\omega_{l+1}|^m = o\left(1\right), \int_{\partial B_\alpha} |D\left(\omega_{l+1} - \omega_{l+1}\left(0\right)\right)|^m = o\left(1\right). \tag{4.30}$$

$$\int_{\partial B_\beta} |D\left(u_n\left(a_n^1 + \lambda_n^1\cdot\right) - \omega_1\left(\infty\right)\right)|^m = o\left(1\right). \tag{4.31}$$

Then

$$\int_{B_R} |D\left(w_n\left(a_n^{l+1} + \lambda_n^{l+1}\cdot\right) - \omega_{l+1}\right)|^m = \int_{A\left(a_n^1, \beta\lambda_n^1, \alpha\lambda_n^{l+1}\right)} |Du_n|^m + o\left(1\right), \tag{4.32}$$

$$\leq \min\left\{\frac{\epsilon_1^m}{2}, \frac{\epsilon_0^m}{2}\right\} + o\left(1\right).$$

If we define v_n on $A\left(a_n^1, \beta\lambda_n^1, \alpha\lambda_n^{l+1}\right)$

$$\begin{aligned} &-\mathrm{div}\left(|Dv_n|^{m-2}Dv_n\right) = 0, \\ &\qquad v_n = u_n, \text{ on } \partial A\left(a_n^1, \beta\lambda_n^1, \alpha\lambda_n^{l+1}\right). \end{aligned} \tag{4.33}$$

Then, from lemma 3.3, we have

$$\int_{A\left(a_n^1, \beta\lambda_n^1, \alpha\lambda_n^{l+1}\right)} |Du_n|^m \leq C\int_{A\left(a_n^1, \beta\lambda_n^1, \alpha\lambda_n^{l+1}\right)} |Dv_n|^m + C\left\|h_n\right\|_{W^{-1,p}}^{m'}. \tag{4.34}$$

In particular,

$$\int_{B_R} \left|D\left(w_n\left(a_n^{l+1} + \lambda_n^{l+1}\cdot\right) - \omega_{l+1}\right)\right|^m \leq C\int_{A\left(a_n^1, \beta\lambda_n^1, \alpha\lambda_n^{l+1}\right)} |Dv_n|^m + o\left(1\right). \tag{4.35}$$

Now, we define f_n and g_n on $A\left(a_n^1, \beta\lambda_n^1, \alpha\lambda_n^{l+1}\right)$ by

$$(4.36) \qquad \begin{aligned} -\mathrm{div}\left(|Df_n|^{m-2}Df_n\right) &= 0, \\ f_n &= u_n - \omega_{l+1}(0), \text{on } \partial B_{\alpha\lambda_n^{l+1}}\left(a_n^1\right) \\ f_n &= u_n - \omega_1(\infty), \text{on } \partial B_{\beta\lambda_n^1}\left(a_n^1\right). \end{aligned}$$

$$(4.37) \qquad \begin{aligned} -\mathrm{div}\left(|Dg_n|^{m-2}Dg_n\right) &= 0, \\ g_n &= \omega_{l+1}(0), \text{ on } \partial B_{\alpha\lambda_n^{l+1}}\left(a_n^1\right) \\ g_n &= \omega_1(\infty), \text{ on } \partial B_{\beta\lambda_n^1}\left(a_n^1\right). \end{aligned}$$

Then, from (4.30)- (4.31) and Proposition 3.2, we have, with $A = A\left(a_n^1, \beta\lambda_n^1, \alpha\lambda_n^{l+1}\right)$,

$$(4.38) \qquad \int_A |Df_n|^m = o(1), \int_A |Dg_n|^m = \frac{|\omega_1(\infty) - \omega_{l+1}(0)|^m}{\left(\log \frac{\alpha\lambda_n^{l+1}}{\beta\lambda_n^1}\right)^{m-1}}.$$

By the minimality of v_n, we know that

$$\int_A |Dv_n|^m \leq \int_A |Df_n|^m + \int_A |Dg_n|^m$$

It follows from (4.38) and $\lambda_n^{l+1}/\lambda_n^1 \to \infty$, for fixed α and β,

$$(4.39) \qquad \int_{A\left(a_n^1, \beta\lambda_n^1, \alpha\lambda_n^{l+1}\right)} |Dv_n|^m = o(1).$$

Now (4.39) and (4.35) imply

$$\int_{B_R} |D\left(w_n\left(a_n^{l+1} + \lambda_n^{l+1}\cdot\right) - \omega_{l+1}\right)|^m = o(1)$$

This completes proof of the claim. We can repeat this argument for w_n near other points in Σ_1 to get bubbles $\omega_j : R^m \to R^k$ and $\left\{a_n^j\right\} \subset \Omega$, $\lambda_n^j \subset R$ for $l+1 \leq j \leq l+l'$ such that (2) in Theorem 1.1 holds and (4.8) holds with l replaced by $l+l'$, $w_n\left(a_n^j + \lambda_n^j\cdot\right) \to \omega_j$ in $W^{1,m}_{\mathrm{loc}}\left(R^m, R^k\right)$. Since each $\omega_i : S^m \to R^k$ has at least energy μ, this bubbling process terminates after $\left[\frac{C}{\mu}\right]$ times, where $C = \lim_n \int_\Omega |Du_n|^m$. Notice that the argument keeps all energy during the bubbling process. The conclusions of theorem follow. ■

Proof of Corollary 1.3. Suppose the conclusion were not true. Then, from theorem 1.1, there exists at least one non-constant m-harmonic map $\omega : R^m \to N$ such that $\lim_n \int_\Omega |Du_n|^m \geq \int_\Omega |Du|^m + \mu$, which contradicts to the assumption.

(2): In this case, we will show that any m-harmonic map ω from S^m to N is constant. In fact, if f is the given convex function on N, then we have the following chain rule (see, Jost [Jj]):

$$(4.30) \qquad \mathrm{div}\left(|D\omega|^{m-2}D(f\circ\omega)\right) \geq c_0|D\omega|^m.$$

Then we integrate this inequality over S^m to conclude that $E_m(\omega) = 0$.

Proof of Theorem 1.2. We start by recalling that on a Riemannian homogeneous manifold (for details, we refer to Helein [Hf] or Toro-Wang [TW]), that there exist q fields Y_α $(1 \le \alpha \le q)$ and q smooth killing vector fields X_α $(1 \le \alpha \le q)$ on N such that for any $y \in N$ and $v \in T_y N$, we have

$$v = \sum_{\alpha=1}^{q} \langle X_\alpha, v\rangle Y_\alpha.$$

In particular

$$|Du_n|^{m-2} Du_n = \sum_{\alpha=1}^{q} |Du_n|^{m-2} \langle Du_n, X_\alpha(u_n)\rangle Y_\alpha(u_n).$$

1/m Note the property of $X_\alpha (1 \le \alpha \le q)$ that for $1 \le \alpha \le q$

$$-\mathrm{div}\left(|Du_n|^{m-2}\langle Du_n, X_\alpha(u_n)\rangle\right) = \langle h_n, X_\alpha(u_n)\rangle. \tag{4.31}$$

Therefore, we can write the m-harmonic equation as

$$\begin{aligned}
-\mathrm{div}\left(|Du_n|^{m-2} Du_n\right) &= \sum_{\alpha=1}^{q} \langle h_n, X_\alpha(u_n)\rangle Y_\alpha(u_n) \\
&\quad + \sum_{\alpha=1}^{q} \langle |Du_n|^{m-2} Du_n, X_\alpha(u_n)\rangle Y_\alpha(u_n) \\
&= \bar{h}_n + \sum_{\alpha=1}^{q} a_\alpha(u_n, Du_n)\, b_\alpha(u_n).
\end{aligned}$$

Since $h_n \to 0$ in $W^{-1,p}$ for some $p > \frac{m}{m-1}$ and X_α, Y_α are smooth in u, it follows that $\bar{h}_n \to 0$ in W^{-1,p^*} for some $p^* > \frac{m}{m-1}$. The conformal invariance of the m-harmonic map equation follows from that of $\int_\Omega |Du|^m$. Therefore, the conditions of Theorem 1.1 are satisfied, and so the conclusions of Theorem 1.2 hold. ■

Proof of Theorem 1.4. In this case, we already observed in §1 that $f = Hu_1 \wedge \cdots \wedge u_m$ can be written as

$$f^i = \sum_{l=1}^{m} a^{il} \cdot \frac{\partial u^{i+1}}{\partial x_l}$$

with $\mathrm{div}\left(a^i\right) = 0$. Moreover, the blow-up equation is conformally invariant. In particular, conditions of Theorem 1.1 are satisfied and hence the conclusions follow. ■

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