Uniqueness of Energy Minimizing Maps
for Almost All Smooth Boundary Data

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Abstract. Here for a smooth bounded Euclidean domain Ω and a smooth Riemannian manifold N, we show that almost every smooth map $\varphi : \partial\Omega \to N$ serves as boundary data for at most one energy minimizing map $u : \Omega \to N$ [Theorem 5.2].

We also obtain some uniform boundary regularity estimates for energy minimizers [Theorem 2.1], which not only are important to our proof, but also imply some other properties of energy minimizers [Corollaries 2.6–2.9].

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1. Introduction

We suppose that Ω is a bounded $C^{3,\alpha}$ domain in $\mathbf{R}^m$ and N is a $C^{4,\alpha}$ compact Riemannian manifold without boundary, and consider φ in $C^{2,\alpha+}(\partial\Omega, N)$, the $\| \ \|_{C^{2,\alpha}}$ closure of $C^3(\partial\Omega, N)$ in $C^{2,\alpha}(\partial\Omega, N)$. We construct a finite measure μ on $C^{2,\alpha+}(\partial\Omega, N)$ which has positive value on each nonempty open subset of $C^{2,\alpha+}(\partial\Omega, N)$, and prove that

For μ–almost all $\varphi \in C^{2,\alpha+}(\partial\Omega, N)$, there exists at most one minimizer of the energy $\int_\Omega |\nabla u|^2 dx$ among $\{u \in L^{1,2}(\Omega, N) : u|_{\partial\Omega} = \varphi\}$.

So in particular, the nonuniqueness set Z consisting of all $\varphi \in C^{2,\alpha+}(\partial\Omega, N)$ each of which serves as boundary data of at least two different energy minimizers is of first category.

It is well known that if u_1, $u_2 \in L^{1,2}(\Omega, N)$ are energy minimizers with same boundary data, and if N is of nonpositive sectional curvature and u_1 and u_2 are homotopic, or if the images of u_1 and u_2 are contained in a same geodesic ball, then $u_1 = u_2$. See [**HR**], [**SR**, Theorem 2.10] and [**JK**]. Without such assumptions, F. Almgren and E. Lieb [**AL**, Theorem 4.1] proved that the boundary data having unique energy minimizers are actually dense in $H^1(\partial\Omega, N)$.

But in general, energy minimizers do not have uniqueness. For example, R. Hardt and F.H. Lin in [**HL2**] gave a $\varphi \in C^{2,\alpha}(\mathbf{S}^2, \mathbf{S}^2)$, which serves as boundary data of two energy minimizers u_1, $u_2 \in L^{1,2}(\mathbf{B}^3, \mathbf{S}^2)$, with u_1 being smooth while u_2 being singular. (See [**AL**].)

To prove this uniqueness theorem, we need some uniform boundary regularity estimates, which are stated in Theorem 2.1. Its proof is based on R. Schoen and K. Uhlenbeck's boundary regularity theorem in [**SU2**, Theorem 2.7] and a compactness argument (Proposition 2.3 and Lemma 2.5). Some different uniform boundary regularity theorems have been proved in [**AL**], [**HL2**, p114] and [**ML2**]. The Lemmas 2.4 and 2.5, Corollaries 2.6–2.9 in §2 may be useful at some other places.

In §3, we prove a uniqueness property (Lemma 3.1) of energy minimizers when they are considered as solutions of the harmonic map equation. This property gives an alternative representation of the set Z, the formula (4.1). Using the uniform boundary regularity estimates in Theorem 2.1, we prove a quantitative density Lemma 4.1 in §4. Then in §5, we construct the measure μ and prove the main result $\mu(Z) = 0$ by applying the density lemma.

Here (in §5) we follow some arguments of F. Morgan [**MF1, MF2**], whose work on the generic uniqueness of minimizing hypersurfaces motivated the present paper. For the case of smoothly immersed minimal surfaces, B. White [**WB**] also gave a description of the nonuniqueness set Z. For smooth harmonic maps one easily obtains similar results. This and associated problems with a fixed singular set are treated in [**HM**] and [**ML1**].

Finally, in §6, we remark that this uniqueness property holds for energy minimizers whose domain may be a general Riemannian manifold.

2. Uniform Boundary Regularity

The main content of this section is the proof of the uniform boundary regularity Theorem 2.1. The Lemmas 2.4–2.5 and Corollaries 2.6–2.9 should be interesting in their own rights. For simplicity of discussion, we assume that N is a fixed compact $C^{2,\alpha}$ Riemannian submanifold of $\mathbf{R}^n$ without boundary, and Ω is any $C^{2,\alpha}$ bounded domain of $\mathbf{R}^m$ with the Euclidean metric. (for a definition of $C^{2,\alpha}$ domain, see §6.2 of [**GT**].)

Definition. For given positive numbers d_0, δ_0 and Λ_0, we define $\mathcal{M} = \mathcal{M}(d_0, \delta_0, \Lambda_0)$ to be the set of all $C^{2,\alpha}$ domains Ω in $\mathbf{R}^m$ satisfying the following conditions:

i. $\mathrm{diam}(\ \Omega\ ) \leq d_0$;

ii. At each point $\omega \in \partial\Omega$, there exists an orthonormal coordinate system $(y^1, \ldots, y^m)$ at ω such that $\dfrac{\partial}{\partial y^m}$ is in the inward unit normal direction $\mathbf{n}(\omega)$ of $\partial\Omega$ at ω. Furthermore, there exists a $C^{2,\alpha}$ function $f : \mathbf{B}^{m-1}(0,\delta_0) \subset \mathbf{R}^{m-1} \to \mathbf{R}$, such that

$$\bar{\Omega} \cap \mathbf{C}^m(\omega,\delta_0) = \{y \in \mathbf{C}^m(0,\delta_0) : y^m \geq f(y')\},$$

$$\|f\|_{C^{2,\alpha}(\mathbf{B}^{m-1}(0,\delta_0))} \leq \Lambda_0,$$

where $\mathbf{C}^m(\omega,\delta_0)$ is the cylinder $\mathbf{B}^{m-1}(0,\delta_0) \times [-\delta_0,\delta_0]$, y' denotes $(y^1,\ldots,y^{m-1})$.

Clearly, $f(0) = |\nabla f(0)| = 0$. It is easy to see that any $C^{2,\alpha}$ bounded domain Ω is in $\mathcal{M} = \mathcal{M}(d_0,\delta_0,\Lambda_0)$ for some suitably chosen positive numbers d_0, δ_0 and Λ_0. Also, one has that $\mathcal{M}(d_0,\delta_0,\Lambda_0) \subset \mathcal{M}(d_1,\delta_1,\Lambda_1)$, if $d_0 \leq d_1, \delta_0 \geq \delta_1$ and $\Lambda_0 \leq \Lambda_1$.

For $\Omega \in \mathcal{M}(d_0,\delta_0,\Lambda_0)$, we denote:

$$\Gamma = \partial\Omega; \quad d(x) = \mathrm{dist}(x,\partial\Omega), \quad \text{for} \quad x \in \mathbf{R}^m;$$

$$\Gamma_\delta = \{x \in \mathbf{R}^m : d(x) \leq \delta\}, \quad \text{for} \quad \delta \geq 0;$$

$$\Gamma_{+\delta} = \Gamma_\delta \cap \bar{\Omega}, \qquad \Gamma_{-\delta} = \Gamma_\delta \setminus \Omega;$$

$$\Omega_r = \Omega \setminus \Gamma_r, \quad \text{if} \quad r \geq 0; \quad \Omega_r = \mathrm{Int}(\Gamma_r \cup \Omega), \quad \text{if} \quad r \leq 0.$$

The definition of $\mathcal{M}$ implies that, for any $\Omega \in \mathcal{M}$, $\partial\Omega$ has an interior neighborhood $\Gamma_{+\delta_0}$ and an exterior neighborhood $\Gamma_{-\delta_0}$; furthermore, one can easily show that, if $\delta_1 = \min\{\frac{\delta_0}{2}, \frac{1}{(m-1)\Lambda_0}\}$, then Ω has an interior ball and an exterior ball of radius δ_1 at each $\omega \in \partial\Omega$. (To check this, one uses the coordinate system $(y^1,\ldots,y^m)$ in the definition of $\mathcal{M}$ at $\omega \in \partial\Omega$, and shows that $\mathbf{B}^m(\pm p,\delta_1)$ are interior and exterior balls of Ω at ω, respectively, where $p = (0,\ldots,0,\delta_1)$.)

Some other properties of $\mathcal{M}$ are stated in Proposition 2.2.

The reason for us to consider such a family $\mathcal{M}$ with specified parameters d_0, δ_0 and Λ_0 is that some quantitative estimates concerning a solution u on $\Omega \in \mathcal{M}$ of some problems are uniform with respect to $\mathcal{M}$ (i.e., not depending on any particular $\Omega \in \mathcal{M}$, but only on d_0, δ_0 and Λ_0 which define $\mathcal{M}$). The Theorem 2.1 below is another such example.

The norm $\| \; \|_{C^{k,\alpha}} = \| \; \|_{C^{k,\alpha}(\partial\Omega)}$ $(0 \leq \alpha \leq 1,\ k = 0,1,2,...)$ on $C^{k,\alpha}(\partial\Omega,N)$ is defined by:

$$\|\varphi\|_{C^{k,\alpha}} = \inf\{\|u\|_{C^{k,\alpha}(\Omega)} : u \in C^{k,\alpha}(\bar{\Omega},\mathbf{R}^n), u = \varphi \text{ on } \partial\Omega\}.$$

For $\varphi \in C^{k,\alpha}(\partial\Omega,N)$, we denote

$$\mathcal{U}_\varphi \equiv \mathcal{U}_{\varphi,\Omega} = \{u \in L^{1,2}(\Omega,N) : u|_{\partial\Omega} = \varphi\}.$$

Theorem 2.1 (Uniform Boundary Regularity). *There exist positive numbers δ and C, depending only on d_0, δ_0, Λ_0, K and N, so that if $\Omega \in \mathcal{M}$, $\varphi \in C^{k,\alpha}(\partial\Omega,N)$ with $\|\varphi\|_{C^{k,\alpha}} \leq K$ ($k = 1$, or 2) and $u \in \mathcal{U}_{\varphi,\Omega}$ is an energy minimizer, then $u \in C^{k,\alpha}(\Gamma_{+\delta},N)$ and*

$$\|u\|_{C^{k,\alpha}(\Gamma_{+\delta})} \leq C.$$

To prove this theorem, we need a compactness property of $\mathcal{M}$ (Proposition 2.3) and a lemma (Lemma 2.5). We start with some basic properties of an domain $\Omega \in \mathcal{M}$.

For $\Omega \in \mathcal{M}$ and $\omega \in \partial\Omega$, let $(y^1, \ldots, y^m)$ be a coordinate system and f be a function in the definition of $\mathcal{M}$. As $|D_{ij}f(0)| \leq \Lambda_0$, the *principal curvatures* $\kappa_1, \ldots, \kappa_{m-1}$ of $\partial\Omega$ at ω, by definition, being the $m-1$ eigenvalues of the Hessian$[D_{ij}f(0)]_{(m-1)\times(m-1)}$ of f at 0, satisfy the following

$$\kappa \equiv \max\{|\kappa_i| : \omega \in \partial\Omega, 1 \leq i \leq m-1\} \leq (m-1)\Lambda_0.$$

The coordinate $(y^1, \ldots, y^{m-1})$ is called *principal coordinate system* at ω if $\dfrac{\partial}{\partial y^i}$ is in the direction of the eigenvector corresponding to κ_i, $1 \leq i \leq m-1$. Under the principal coordinate system, Hessian$[D_{ij}f(0)]_{(m-1)\times(m-1)}$ =diag$[\kappa_1, \ldots, \kappa_{m-1}]$.

Since Ω has uniformly interior and exterior balls of radius δ_1, we have that, for each $x \in \Gamma_{\delta_1}$, there exists a unique $\omega \in \partial\Omega$, denoted by $\pi(x)$, satisfying $d(x) = \mathrm{dist}(x, \partial\Omega) = |x - \pi(x)|$. Denote $\mathbf{n}(\omega)$ the inward unit normal direction of $\partial\Omega$ at $\omega \in \partial\Omega$. We have the following (cf. [**GT**, Appendix] and [**AW**, Lemma 2.2].)

Proposition 2.2. *Let Ω be in $\mathcal{M}$, then*

a. $\mathbf{n} \in C^{1,\alpha}(\partial\Omega, \mathbf{R}^m)$, $\pi \in C^{1,\alpha}(\Gamma_{\delta_1}, \partial\Omega)$ *and* $d \in C^{2,\alpha}(\Omega)$.

b. *Let* $\Gamma_{t,s} = \Gamma_{+t} \cup \Gamma_{-s}$ *for* $s, t \in [0, \delta_1]$, *then* $\Gamma_{t,s} = \{\omega + r\mathbf{n}(\omega) : \omega \in \partial\Omega \ -s \leq r \leq t\}$. *The map* $\mathbf{F}(\omega, r) = \omega + r\mathbf{n}(\omega)$ *is a* $C^{1,\alpha}$ *diffeomorphism between* $\partial\Omega \times [-s, t]$ *and* $\Gamma_{t,s}$, *and for each* $r \in [-\delta_1, \delta_1]$, $\mathbf{F}_r(\cdot) = \mathbf{F}(\cdot, r)$ *is a* $C^{1,\alpha}$ *diffeomorphism between* $\partial\Omega$ *and* $\partial\Omega_r$.

c. *If* $\delta_2 = \min\{\delta_1, [(2m+2)\kappa]^{-1}\}$, *then the Jacobi of* $\mathbf{F}_r$ *satisfies*

$$|J(\mathbf{F}_r)(\omega) - 1| \leq 2(m-1)\kappa|r|, \tag{2.1}$$
$$|\mathrm{vol}(\Gamma_{\pm t}) - \mathrm{area}(\partial\Omega)t| \leq 2(m-1)\kappa \ \mathrm{area}(\partial\Omega)t^2,$$

for $(\omega, r) \in \partial\Omega \times [-\delta_2, \delta_2]$ *and* $0 \leq t \leq \delta_2$, *respectively.*

d. *If* $0 \leq s, t \leq \delta_2$ *and* $u \in L^{1,2}(\Gamma_{t,s}, N)$, *then*

$$\int_{\Gamma_{t,s}} |\nabla u|^2 dx \leq (1 + c_{st}) \int_{\partial\Omega} \int_{-s}^{t} \left(|\nabla_\omega u|^2 + \left|\frac{\partial u}{\partial r}\right|^2 \right)(\omega, r) d\omega dr, \tag{2.2}$$
$$\left[\geq (1 - c_{st}) \int_{\partial\Omega} \int_{-s}^{t} \left(|\nabla_\omega u|^2 + \left|\frac{\partial u}{\partial r}\right|^2 \right)(\omega, r) d\omega dr. \right]$$

where on the right hand side, $u(\omega, r) = u(\omega + r\mathbf{n}(\omega))$ *and* $c_{st} = 2(m+1)\kappa \max\{s, t\}$.

Proof. For $x \in \Gamma_{\delta_1}$, let $\omega = \pi(x) \in \partial\Omega$. Thus, $x = \omega + \mathbf{n}(\omega)r$, where $r = \pm d(x)$ (depending on $x \in \Gamma_{+\delta_1}$ or $x \in \Gamma_{-\delta_1}$).

For a fixed $x_0 \in \Gamma_{\delta_1}$, denote $\omega_0 = \omega(x_0)$. We may assume that $\omega_0 = 0$ and that $T_{\omega_0}(\partial\Omega) = \mathbf{R}^{m-1} \times \{0\}$ with the principal coordinate system. By the definition of $\mathcal{M}$, $\partial\Omega \cap \mathbf{C}^m(0,\delta_0)$ is the graph of a $C^{2,\alpha}$ function $f : \mathbf{B}^{m-1}(0,\delta_0) \to \mathbf{R}$. Define a map $\mathbf{g} : \mathbf{B}^{m-1}(0,\delta_0) \times [-\delta_0,\delta_0] \to \mathbf{R}^m$ by

$$x = \mathbf{g}(y',r) = \omega + \mathbf{n}(\omega)r, \text{ where } \omega = (y', f(y')), \tag{2.3}$$

then $\mathbf{g}$ is $C^{1,\alpha}$ on $\mathbf{B}^{m-1}(0,\delta_0) \times [-\delta_0,\delta_0]$ and its Jacobi matrix of at $(0,r)$ is

$$[D\mathbf{g}] = \mathrm{diag}[1-\kappa_1 r, \ldots, 1-\kappa_{m-1}r, 1];$$

therefore, $\mathbf{g}$ is regular when $|r| \le \delta_1$ ($\delta_1 = \min\{\frac{\delta_0}{2}, \frac{1}{(m-1)\Lambda_0}\}$). Particularly, $\mathbf{g}$ is regular at $(0, d(x_0))$. It follows from the inverse mapping theorem that, for x in a neighborhood of $x_0 = \mathbf{g}(0, d(x_0))$, the map $y'(x)$ is $C^{1,\alpha}$. Consequently, the maps $\omega = (y', f(y')) = \pi(x)$ and $r = d(x)$ are $C^{1,\alpha}$. Since $Dd(x) = \mathbf{n}(y(x)) \in C^{1,\alpha}$, one has that $r = d(x)$ is actually $C^{2,\alpha}$. This shows (a).

The proof of (b) is easy, since for each $x \in \Gamma_{\pm\delta_1}$, one has that $x = \omega \pm r\mathbf{n}(\omega)$ with $r = d(x)$. That the map $\mathbf{g}$ is regular when $|r| \le \delta_1$ implies that $\mathbf{F}$ and $\mathbf{F}_r$ are diffeomorphisms.

For a proof of (c), define $\mathbf{G} : \mathbf{B}^{m-1}(0,\delta_0) \times \mathbf{R} \to \partial\Omega \times \mathbf{R} \subset \mathbf{R}^m \times \mathbf{R}$ by $\mathbf{G}(y',r) = (y', f(y'), r)$. Then

$$[D\mathbf{G}](0,r) = \begin{pmatrix} I_{(m-1)\times(m-1)} & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}_{m\times(m+1)}.$$

Since $\mathbf{F} = \mathbf{g} \circ \mathbf{G}^{-1}$ when $|r| \le \delta_2$, we have

$$J(\mathbf{F})(\omega_0, r) = \prod_1^{m-1} \frac{1}{1-\kappa_i|r|} \le \frac{1}{1-(m-1)\kappa|r|} \le 1 + 2(m-1)\kappa|r|. \tag{2.4}$$
$$\left[\ge \frac{1}{1+(m-1)\kappa|r|} \ge 1 - 2(m-1)\kappa|r|.\right]$$

From this and an area formula (e.g., see §8 of [**SL**].), we get

$$(1-2(m-1)\kappa t)\mathrm{area}(\partial\Omega)t \le \mathrm{vol}(\Gamma_{\pm t}) \le (1+2(m-1)\kappa t)\mathrm{area}(\partial\Omega)t,$$

for $t \in [0, \delta_2]$. This shows (c).

If $u \in L^{1,2}(\Gamma_{t,s}, N)$, then by the change of variables $x \to (y', r)$ defined by (2.3), we have

$$
\begin{aligned}
|\nabla u|^2(x_0) &= \sum_{\alpha=1}^{n}\sum_{i=1}^{m}\left(\frac{\partial u_\alpha}{\partial x^i}\right)^2 \\
&= \sum_{\alpha=1}^{n}\sum_{i=1}^{m-1}\frac{1}{(1-\kappa_i r)^2}\left(\frac{\partial u_\alpha}{\partial y^i}\right)^2 + \sum_{\alpha=1}^{n}\left(\frac{\partial u_\alpha}{\partial r}\right)^2 \\
&\leq \frac{1}{(1-2\kappa|r|)}\sum_{i=1}^{m-1}\left|\frac{\partial u}{\partial y^i}\right|^2 + \left|\frac{\partial u}{\partial r}\right|^2 \\
&\leq \frac{1}{(1-2\kappa|r|)}\left(|\nabla_\omega u|^2 + \left|\frac{\partial u}{\partial r}\right|^2\right)(\omega_0, r).
\end{aligned}
\tag{2.5}
$$

This estimate holds at any point $x = \omega + r\mathbf{n}(\omega) \in \Gamma_{+t}$, which corresponds to $(\omega, r) \in \partial\Omega \times [-s, t]$. So by (2.4) and (2.5)

$$
\begin{aligned}
\int_{\Gamma_{t,s}} |\nabla u|^2 dx &\leq \frac{1}{1-2\kappa|r|}\int_{\partial\Omega}\int_{-s}^{t}\left(|\nabla_\omega u|^2 + \left|\frac{\partial u}{\partial r}\right|^2\right)(\omega, r) J(\mathbf{F}) d\omega dr \\
&\leq (1 + c_{st})\int_{\partial\Omega}\int_{-s}^{t}\left(|\nabla_\omega u|^2 + \left|\frac{\partial u}{\partial r}\right|^2\right)(\omega, r) d\omega dr. \\
&\left[\geq (1 - c_{st})\int_{\partial\Omega}\int_{-s}^{t}\left(|\nabla_\omega u|^2 + \left|\frac{\partial u}{\partial r}\right|^2\right)(\omega, r) d\omega dr.\right]
\end{aligned}
$$

Thus we finish the proof of Proposition 2.2. ■

Remark. If one replaces $C^{2,\alpha}$ by $C^{k,\alpha}$ or C^k with $k \geq 2$ in the definition of $\mathcal{M}$, then the discussions above on $\Omega \in \mathcal{M}$ still hold. In particular, Proposition 2.2 holds with $C^{2,\alpha}$ replaced by $C^{k,\alpha}$ or C^k, and $C^{1,\alpha}$ replaced by $C^{k-1,\alpha}$ or C^{k-1}, respectively.

Since each $\Omega \in \mathcal{M}$ is contained in a ball of radius d_0, we will not lose generality in considering the subset $\mathcal{M}_0 = \{\Omega \in \mathcal{M} : \Omega \subset \bar{\mathbf{B}}^m(0, d_0)\}$. For $\mathcal{M}_0$, we have

Proposition 2.3. *For any sequence $\{\Omega^i\}$ in $\mathcal{M}_0$, there exists a subsequence $\{\Omega^k\}$ of $\{\Omega^i\}$ and a domain $\Omega \in \mathcal{M}_0$ so that $\{\Omega^k\}$ converges to Ω in the following $C^{1,\alpha/2}$ sense: There are $C^{1,\alpha}$ diffeomorphisms $\mathbf{f}_k : \bar{\Omega} \to \bar{\Omega}^k$ such that $\mathbf{f}_k \to \mathbf{I}_\Omega$ in $C^{1,\alpha/2}$, where $\mathbf{I}_\Omega$ is the identity on Ω.*

Proof. We will use the following definitions: For $A, B \subset \mathbf{R}^m$ and $\varepsilon > 0$, one has the *Hausdorff distance* $d_H(A, B) = \inf\{\varepsilon > 0 : A \subset B_\varepsilon,\ B \subset A_\varepsilon\}$, where $A_\varepsilon = \{x \in$

$\mathbf{R}^m : \operatorname{dist}(x, A) \le \varepsilon\}$. Then for a sequence A^1, A^2, ... of subsets of $\mathbf{R}^m$, $A = d_H$-$\lim A^i$ iff $d_H(A^i, A) \to 0$, as $i \to \infty$. We also define $\limsup A^i = \cap_{j=1}^{\infty} \cup_{i=j}^{\infty} (A^i)_{i^{-1}}$, $\liminf A^i = \cup_{j=1}^{\infty} \cap_{i=j}^{\infty} (A^i)_{i^{-1}}$. Obviously, $x \in \limsup A^i$ iff there exists a subsequence $\{j\}$ of $\{i\}$ and $x_j \in A^j$ such that $x_j \to x$ as $j \to \infty$; that $A = d_H$-$\lim A^i$ implies that $A = \limsup A^i$. Finally we say that finite points $x_1, \dots, x_q$ form a *δ-net* in A if $\bar{A} \subset \cup_{i=1}^{q} \mathbf{B}(x_i, \delta)$.

Let $\delta = \delta_1/4$. Suppose that $x_1, \dots, x_{q_0}$ form a δ-net in $\mathbf{B}^m(0, d_0)$ (q_0 depends only d_0 and δ_0). Suppose that $\{\Omega^i\}$ is a sequence in $\mathcal{M}_0$. We consider the sequence $\{\partial\Omega^i\}$ of their boundaries. Let p_1, $1 \le p_1 \le q_0$, be the minimal integer such that $\limsup \partial\Omega^i \cap \mathbf{B}^m(x_{p_1}, \delta) \neq \emptyset$, say containing a point $\omega_{p_1}^0$. Then we have a subsequence $\{j\}$ of $\{i\}$ and point $\omega_{p_1}^j \in \partial\Omega^j$ for each j, so that $|\omega_{p_1}^j - \omega_{p_1}^0| \to 0$ as $j \to \infty$. Let p_2, $p_1 < p_2 \le q_0$ be the next minimal integer such that $\limsup \partial\Omega^j \cap \mathbf{B}^m(x_{p_2}, \delta) \neq \emptyset$. As above, we have a point $\omega_{p_2}^0$ in it, a subsequence $\{k\}$ of $\{j\}$ and $\omega_{p_2}^k \in \partial\Omega^k$ for each k, so that $|\omega_{p_2}^k - \omega_{p_2}^0| \to 0$ as $k \to \infty$. Repeat this procedure successively up to q_0, we get an integer q, $1 \le q \le q_0$, and q of the covering balls, say $\mathbf{B}^m(x_1, \delta), \dots, \mathbf{B}^m(x_q, \delta)$ after re-ordering, and also a subsequence of $\{i\}$, say $\{j\}$, so that for each $q < p \le q_0$, $\limsup \partial\Omega^j \cap \mathbf{B}^m(x_p, \delta) = \emptyset$, but for each $1 \le p \le q$, $\limsup \partial\Omega^j \cap \mathbf{B}^m(x_p, \delta)$ contains at least one point ω_p^0 and there exists a corresponding sebquence $\omega_p^j \in \partial\Omega^j$ satisfying $|\omega_p^j - \omega_p^0| \to 0$ as $j \to \infty$. We may assume that $|\omega_p^j - \omega_p^0| \le \delta$ for all j and all $p = 1, \dots, q$, since it is true for j large.

As $\mathbf{B}^m(x_p, \delta) \subset \mathbf{B}^m(\omega_p^0, 2\delta)$, $1 \le p \le q$, one has that $\omega_1^0, \dots, \omega_q^0$ form a 2δ-net in $\limsup \partial\Omega^j$. This implies that we may assume that $\omega_1^j, \dots, \omega_p^j$ form a 3δ-net in $\partial\Omega^j$ for each j, since $|\omega_p^j - \omega_p^0| \to 0$ implies this is true for j large.

Let $\mathbf{n}^j$ be the inward unit normal direction of $\partial\Omega^j$. Considering $\{\mathbf{n}^j(\omega_1^j)\}$ as a sequence of unit vectors of $\mathbf{R}^m$, we may assume that $\mathbf{n}^j(\omega_1^j) \to \mathbf{n}^0$ (as $j \to \infty$, by passing to a subsequence). This means that the sequence $\{T_{\omega_1^j}(\partial\Omega^j)\}$ of tangent planes converges to $T_{\omega_1^0}$, a plane passing through ω_1^0 and with $\mathbf{n}^0$ being its normal direction. In other words, one may take an affine map $\mathbf{T}_j$ (for each j) from $\mathbf{R}^m$ to $\mathbf{R}^m$ mapping ω_1^0 to ω_1^j and $T_{\omega_1^0}$ to $T_{\omega_1^j}(\partial\Omega^j)$, and perserving the orientations, so that $\mathbf{T}_j \to \mathbf{I}$ in C^∞.

Now recall the definition of $\mathcal{M}$, that $\mathbf{C}^m(\omega_1^j, \delta_1) \cap \partial\Omega^j$ is the graph of a function $f^j : \mathbf{B}^{m-1}(\omega_1^j, \delta_1) \to \mathbf{R}$ with $\|f^j\|_{C^{2,\alpha}(\mathbf{B}^{m-1}(\omega_1^j, \delta_1))} \le \Lambda_0$. So $\mathbf{T}_j^{-1}(\partial\Omega^j) \cap \mathbf{C}(\omega_1^0, \delta_1)$ is the graph of $f^j \circ \mathbf{T}_j$ over $T_{\omega_1^0} \cap \mathbf{B}^{m-1}(\omega_1^0, \delta_1)$; furthermore, $\|f^j \circ \mathbf{T}_j\|_{C^{2,\alpha}(\mathbf{B}^{m-1}(\omega_1^0, \delta_1))} \le \Lambda_0$. By Arzela's theorem, there is a subsequence $\{k\}$ of $\{j\}$ and a $C^{2,\alpha}$ function $f^0 : \mathbf{B}^{m-1}(\omega_1^0, \delta_1) \to \mathbf{R}$ such that $f^k \circ \mathbf{T}_k \to f^0$ in $C^{2,\alpha/2}$. Therefore, (2.6)

$$\mathbf{C}^m(\omega_1^0, \delta_1) \cap \operatorname{graph}(f^0) = d_H\text{-}\lim \mathbf{C}^m(\omega_1^k, \delta_1) \cap \partial\Omega^k = d_H\text{-}\lim \mathbf{T}_k^{-1}(\partial\Omega^k) \cap \mathbf{C}^m(\omega_1^0, \delta_1). \tag{2.6}$$

The latter equality comes from that $\mathbf{T}_k \to \mathbf{I}$.

In fact we have $C^{2,\alpha}$ diffeomorphisms

$$\mathbf{f}_k : \{y \in \mathbf{C}^m(\omega_1^0, \delta_1) : y^m \geq f^0(y')\} \to \{y \in \mathbf{B}^{m-1}(\omega_1^k, \delta_1) \times \mathbf{R} : y^m \geq f^0(y')\}$$

which is defined by

$$\mathbf{f}_k(y) = \mathbf{T}_k(y', y^m - f^0(y')) + (0, \ldots, 0, f^k(\mathbf{T}_k(y', 0)),$$

and satisfies that $\mathbf{f}_k \to \mathbf{I}$ in $C^{2,\alpha/2}$.

Repeat the same discussion for ω_2^0, and then for ω_3^0, successively up to ω_q^0, we get a subsequence of $\{j\}$, say $\{k\}$, so that d_H-$\lim \partial\Omega^k$ is locally graphs of $C^{2,\alpha}$ functions; therefore, d_H-$\lim \partial\Omega^k$ is a closed $C^{2,\alpha}$ hypersurface. Let Ω be the unique bounded domain such that d_H-$\lim \partial\Omega^k$ is its boundary $\partial\Omega$ and $\mathbf{n}^0$ is its inward unit normal direction at ω_1^0. The above discussions show the following: $\Omega \in \mathcal{M}_0$; Ω^k are locally $C^{2,\alpha}$ diffeomorphic to Ω near their boundaries; $\partial\Omega = d_H$-$\lim \partial\Omega^k$ from (2.6) and that $\omega_1^j, \ldots, \omega_p^j$ form a 3δ-net in $\partial\Omega^j$ for each j. Therefore, we have d_H-$\lim \Omega^k = \Omega$.

Now we construct $C^{1,\alpha}$ diffeomorphisms $\mathbf{f}_k : \bar\Omega \to \bar\Omega^k$ (for k large) satisfying that $\mathbf{f}_k \to \mathbf{I}$ in $C^{1,\alpha/2}$. Let k be so large that $d_H(\partial\Omega, \partial\Omega^k) \leq \delta_1$, then $\partial\Omega^k \subset \Gamma_{\delta_1}$ (where $\Gamma = \partial\Omega$). Let $\pi : \Gamma_{\delta_1} \to \partial\Omega$ be the projectional map. Then $\pi|_{\partial\Omega^k}$ are $C^{1,\alpha}$ diffeomorphisms between $\partial\Omega^k$ and $\partial\Omega$ when k is large. (cf. Proposition 2.2(b)) Denote $\pi_k = \left(\pi|_{\partial\Omega^k}\right)^{-1}$. By passing to a subsequence, we may assume that $\pi_k \to \mathbf{I}_{\partial\Omega}$ in $C^{1,\alpha/2}$. Let $\eta(r,s)$ be a C^2 function on $[0, 2\delta] \times [-2\delta, 2\delta]$, such that $\eta(2\delta, s) = 2\delta$, and $\eta(0, s) = s$, $\big($e.g., $\eta(r,s) = r + s(1 - r/2\delta)^3\big)$. Define

$$\mathbf{f}_k(x) = \begin{cases} \omega + \eta(r, d_k(\omega))\mathbf{n}(\omega), & x = \omega + r\mathbf{n}(\omega),\ 0 \leq r \leq \delta, \\ x, & x \in \Omega_\delta, \end{cases}$$

where $d_k(\omega) = \mathbf{n}(\omega) \bullet (\pi_k(\omega) - \omega) = \pm\mathrm{dist}(\omega, \pi_k(\omega) \in C^{1,\alpha}(\partial\Omega)$. Now it is easy to see that $\mathbf{f}_k$ are $C^{1,\alpha}$ diffeomorphisms and $\to \mathbf{I}_\Omega$ in $C^{1,\alpha/2}$. ■

Remark. In Proposition 3.1, if g_0 is the Euclidean metric on $\mathbf{R}^m$ and $g_k = \mathbf{f}_k{}^* g_0$ is the pull-back metric of g_0 via $\mathbf{f}_k : \Omega \to \Omega^k$, then clearly $g_k \to g_0$ in $C^{\alpha/2}$.

For a fixed $\Omega \in \mathcal{M}$ and a positive number $\Lambda_1 > 1$, consider the family $\mathcal{G}_{\Lambda_1}$ consisting of all C^0 metrics $g = g_{ij} dx_i\, dx_j$ on Ω satisfying the following

$${\Lambda_1}^{-1}(\delta_{ij}) \leq (g_{ij}) \leq \Lambda_1(\delta_{ij}).$$

Then $L^{1,2}(\Omega, N)$ are invariant with respect to $g \in \mathcal{G}_{\Lambda_1}$. We will denote $E(u,g)$ the energy of $u \in L^{1,2}(\Omega, N)$ with respect to metric g, and $E(u) \equiv E(u, g_0)$.

There is a positive number $\tau > 0$ so that for each $x \in N_\tau = \{x \in \mathbf{R}^n : \operatorname{dist}(x, N) < \tau\}$ there exists a unique $\pi(x) = \pi_N(x) \in N$ satisfying $|x - \pi(x)| = \operatorname{dist}(x, N)$ and

$$\|D\pi(x) - P_{\pi(x)}\| \le C_0 |x - \pi(x)|, \qquad (2.7)$$
$$|D\pi|(x) \le C_1$$

for $x \in N_\tau$. Where C_0 and C_1 are constants depending only on N, and $P_{\pi(x)}$ is the orthogonal projection of $\mathbf{R}^n$ onto $T_{\pi(x)}(N)$. (See [**HL1**] p558.)

As we will apply the boundary regularity theorem in [**SU2**, Theorem 2.7] many times, we restate it as follows: (See [**HKL**] for the cases of lower order regularity.)

Boundary Regularity Theorem in [SU2]. *Suppose Ω is a bounded $C^{2,\alpha}$ domain in $\mathbf{R}^m$ and N is a compact $C^{2,\alpha}$ Riemannian submanifold of $\mathbf{R}^n$ without boundary. If $\varphi \in C^{k,\alpha}(\partial\Omega, N)$ ($k = 1$ or 2), and $u \in L^{1,2}(\Omega, N)$ is an energy minimizer with $u = \varphi$ on $\partial\Omega$. Then u is $C^{k,\alpha}$ in a neighborhood of $\partial\Omega$.*

Lemma 2.4. *Suppose Ω is a bounded $C^{2,\alpha}$ domain in $\mathbf{R}^m$ and N is a compact $C^{2,\alpha}$ Riemannian submanifold of $\mathbf{R}^n$ without boundary. If $\mathcal{U}_\varphi \ne \emptyset$ for some $\varphi \in C^{1,\alpha}(\partial\Omega, N)$, then $\mathcal{U}_\psi \ne \emptyset$ for any $\psi \in C^{1,\alpha}(\partial\Omega, N)$ with $\|\psi - \varphi\|_{C^0} \le \tau$.*

In fact, if $u \in \mathcal{U}_\varphi$ is an energy minimizer, and $\Lambda_1 > 1$, then there exist positive numbers δ_3 and C, depending only on $\|\varphi\|_{C^1(\partial\Omega)}$, u, Λ_1, Ω and N, so that for any $0 < t \le \delta_3$ and $g \in \mathcal{G}_{\Lambda_1}$, there is a $v \in \mathcal{U}_\psi$ satisfying

$$E(v, g) \le E(u, g) + C\left[t + t\|\varphi - \psi\|^2_{C^1(\partial\Omega)} + \frac{1}{t}\|\varphi - \psi\|^2_{C^0(\partial\Omega)}\right]. \qquad (2.8)$$

In particular, for $g \in \mathcal{G}_{\Lambda_1}$, there is a $v \in \mathcal{U}_\psi$ satisfying

$$E(v, g) \le E(u, g) + C\|\varphi - \psi\|_{C^0(\partial\Omega)},$$

where C depends additionally on $\|\psi\|_{C^1(\partial\Omega)}$. (cf. Corollary 2.8.)

Proof. By the boundary regularity theorem in [**SU2**], being an energy minimizer in $\mathcal{U}_\varphi$, u is $C^{1,\alpha}$ in $\Gamma_{+2\delta_3}$ for some $0 < \delta_3 \le \delta_2/2$. For $0 < t \le \delta_3$ and ψ as stated, we define

$$v(x) = \begin{cases} \pi_N\big[(1 - \eta(r))\psi(\omega) + \eta(r)\varphi(\omega)\big] & \text{for } x = \omega + r\mathbf{n}(\omega) \in \Gamma_{+t}; \\ u\big(\omega + 2t\eta(r - t)\mathbf{n}(\omega)\big) & \text{for } x = \omega + r\mathbf{n}(\omega) \in \Omega_t \setminus \Omega_{2t}; \\ u(x) & \text{for } x \in \Omega_{2t}, \end{cases}$$

where η is an increasing and smooth cutoff function satisfying that $\eta(r) = 0$ when $r \le 0$, $\eta(r) = 1$ when $r \ge t$ and $\eta'(r) \le 2/t$. Note that $\pi_N[\cdots]$ is well-defined as $(1-\eta(r))\psi(\omega) + \eta(r)\varphi(\omega) \in N_\tau$. Clearly $v \in \mathcal{U}_\psi$; therefore, $\mathcal{U}_\psi \neq \emptyset$.

Now suppose that $g \in \mathcal{G}_{\Lambda_1}$. For $x \in \Gamma_{+t}$, by (2.5) and (2.7), we have

$$\begin{aligned}
|\nabla v(x)|_g^2 &= g^{ij}\frac{\partial v^\alpha}{\partial x^i}\frac{\partial v^\alpha}{\partial x^j} \le \Lambda_1 |\nabla v|^2(x) \\
&\le C_1\Lambda_1\big(1 + 2(m+1)\kappa t\big)\left[\frac{4}{t^2}|\psi(\omega) - \varphi(\omega)|^2 + (|\nabla_\omega \psi(\omega)| + |\nabla_\omega \varphi(\omega)|)^2\right] \\
&\le C_2\left[\frac{1}{t^2}\|\psi - \varphi\|^2_{C^0(\partial\Omega)} + \|\psi - \varphi\|^2_{C^1(\partial\Omega)} + \|\varphi\|^2_{C^1(\partial\Omega)}\right].
\end{aligned}$$

Similarly, for $x \in \Omega_t \setminus \Omega_{2t}$, we have

$$\begin{aligned}
|\nabla v(x)|_g^2 &\le C_3\left[\left|\frac{\partial u}{\partial \mathbf{n}}(\omega + 2t\eta\mathbf{n}(\omega))\right|^2 + \big(m + t|\nabla_\omega \mathbf{n}(\omega)|\big)|\nabla u(\omega + 2t\eta\mathbf{n}(\omega))|^2\right] \\
&\le C_4\|\nabla u\|^2_{C^1(\Omega_{2\delta_3})}.
\end{aligned}$$

Then (2.8) is obtained by using the formulae (2.1) and (2.2). The particular case is obtained by choosing $t = \min\{\|\varphi - \psi\|_{C^0(\partial\Omega)}, \delta_3\}$. ■

Lemma 2.5. *Suppose that Ω and N are the same as in Lemma 2.4; $\{g_i\}$ is a sequence of metrics on Ω converging to the Euclidean metric g_0 (on Ω) in C^0. If $u_i \in L^{1,2}(\Omega, N)$ is an energy minimizer with respect to g_i, $u_i|_{\partial\Omega} = \varphi_i \in C^{1,\alpha}(\partial\Omega, N)$ and $\|\varphi_i\|_{C^{1,\alpha}(\partial\Omega)}$ is bounded. Then we have*

- a. *If u_i converges to u weakly in $L^{1,2}(\Omega, N)$, then u is an energy minimizer with $u = \varphi$ on $\partial\Omega$ for some $\varphi \in C^{1,\alpha}(\partial\Omega, N)$.*
- b. *Generally, any such a sequence $\{u_i\}$ of energy minimizers has a subsequence strongly converging to an energy minimizer.*

Proof of (a). Since $\|\varphi_i\|_{C^{1,\alpha}(\partial\Omega)}$ is bounded, we may assume (by passing to a subsequence) that $\varphi_i \to \varphi$ in C^1 for some $\varphi \in C^{1,\alpha}(\partial\Omega, N)$; furthermore, $u = \varphi$ on $\partial\Omega$ by trace theorem.

Suppose for the sake of contradiction that u is not an energy minimizer, we then take an energy minimizer $v \in \mathcal{U}_\varphi$, which satisfies

$$E(v, g_0) \le E(u, g_0) - \varepsilon, \qquad \text{for some} \qquad \varepsilon > 0.$$

Applying Lemma 2.4 with i being so large that $\|\varphi_i - \varphi\|_{C^0(\partial\Omega)} \le \min\{\varepsilon/4C, \tau\}$ and $g_i \in \mathcal{G}_{\Lambda_1}$, we get a $v_i \in \mathcal{U}_{\varphi_i}$ which satisfies

$$E(v_i, g_i) \le E(v, g_i) + \frac{\varepsilon}{4}.$$

Since $g_i \to g_0$ in C^0, we have, for i large,

$$E(v, g_i) \leq E(v, g_0) + \frac{\varepsilon}{4}.$$

On the other hand, by lower semicontinuity, we have, for i large,

$$E(u, g_0) \leq E(u_i, g_i) + \frac{\varepsilon}{4}.$$

Combining all these inequalities together, we get

$$E(v_i, g_i) \leq E(u_i, g_i) - \frac{\varepsilon}{4}.$$

This contradicts to the minimality of u_i. So u has to be an energy minimizer.

Now for any $\varepsilon > 0$, applying Lemma 2.4 again as above, we have $v_i \in \mathcal{U}_{\varphi_i}$ for large i, so that

$$E(v_i, g_i) \leq E(u) + \varepsilon.$$

So $E(u_i, g_i) \leq E(u) + \varepsilon$ by minimality of u_i; therefore, $E(u_i, g_i) \to E(u)$. This combined the weak convergence of u_i, gives that $u_i \to u$ strongly.

Proof of (b). As above, we can take a subsequence $\{\varphi_j\}$ of $\{\varphi_i\}$ such that $\varphi_j \to \varphi$ in $C^1(\partial\Omega)$ as $j \to \infty$. We first notice that $\mathcal{U}_\varphi \neq \emptyset$. In fact, since $\mathcal{U}_{\varphi_j,\Omega} \neq \emptyset$, Lemma 2.4 implies that $\mathcal{U}_\varphi \neq \emptyset$ when j is so large that $\|\varphi_j - \varphi\|_{C^0(\partial\Omega)} \leq \tau$. Let u be an energy minimizer in $\mathcal{U}_\varphi$

Now we prove that $E(u_j, g_j)$ is bounded. Again applying Lemma 2.4 with i being so large that $\|\varphi_i - \varphi\|_{C^0(\partial\Omega)} \leq \min\{1/C, \tau\}$ and $g_i \in \mathcal{G}_2$, we get a $v_i \in \mathcal{U}_{\varphi_i}$ which satisfies

$$E(v_j, g_j) \leq E(u, g_j) + 1 \leq 2mE(u, g_0) + 1.$$

By minimality of u_j, $E(u_j, g_j) \leq 2mE(u, g_0) + 1$, which is bounded. Therefore, u_j has a subsequence u_k weakly converging to some $u_0 \in \mathcal{U}_\varphi$. Now Part a) implies that u_0 is an energy minimizer and the convergence is strong. ∎

A direct corollary to Lemma 2.5 is the following stability of energy minimizers. (cf. Corollaries 2.8 and 3.3) For the cases $m = 3$ and $N = \mathbf{S}^2$, R. Hardt and F. Lin proved a much stronger stability theorem in [**HL2**, p113].

Corollary 2.6. *Let Ω and N the same as in Lemma 2.4 and $\varphi \in C^{1,\alpha}(\partial\Omega, N)$ with $\|\varphi\|_{C^{1,\alpha}} \leq K$ for some $K > 0$. Given any $\varepsilon > 0$, there exists an $\eta > 0$ (depending on*

Ω, N, φ, K *and* ε*) so that if* $\psi \in C^{1,\alpha}(\partial\Omega, N)$ *with* $\|\psi\|_{C^{1,\alpha}} \le K$ *and* $\|\psi - \varphi\|_{C^0} \le \eta$*, and* $v \in \mathcal{U}_\psi$ *is an energy minimizer, then there is an energy minimizer* $u \in \mathcal{U}_\varphi$ *satisfying*

$$\int_\Omega |\nabla u - \nabla v|^2 \, dx \le \varepsilon.$$

Proof. Suppose for the sake contradiction that this is not true, we then have an $\varepsilon_0 > 0$ and a sequence $\psi_i \in C^{1,\alpha}(\partial\Omega, N)$ with $\|\psi_i\|_{C^{1,\alpha}} \le K$ and a sequence $v_i \in \mathcal{U}_{\psi_i}$ of energy minimizers satisfying $\|\psi_i - \varphi\|_{C^0} \to 0$, but for *each* i and *any* energy minimizer $u \in \mathcal{U}_\varphi$,

$$\int_\Omega |\nabla u - \nabla v_i|^2 dx \ge \varepsilon_0. \tag{2.9}$$

As $\|\psi_i\|_{C^{1,\alpha}}$ is bounded, we may apply Lemma 2.5 with $\Omega^i = \Omega$ to get a subsequence $\{v_j\}$ of $\{v_i\}$ so that $v_j \to v$ strongly in $L^{1,2}(\Omega, N)$, where v is also an energy minimizer. As $\psi_j \to \varphi$ in C^0, we have $v = \varphi$ on $\partial\Omega$ by trace theorem. So v is an energy minimizer in $\mathcal{U}_\varphi$ not satisfying (2.9), a contradiction. ■

Proof of the Theorem 2.1: Suppose for the sake of contradiction that the conclusion is not true, then there exist a sequence of domains $\{\Omega^i\}$ in $\mathcal{M}$, a sequence of boundary data $\{\varphi_i\}$, $\varphi_i \in C^{k,\alpha}(\partial\Omega^i, N)$ with $\|\varphi_i\|_{C^{k,\alpha}} \le K$, a sequence of positive numbers $\{\delta_i\}$, $\delta_i \downarrow 0$, and a sequence of energy minimizers $\{u_i\}$, $u_i \in L^{1,2}(\Omega^i, N)$, $u_i = \varphi_i$ on $\Gamma^i = \partial\Omega^i$, such that

$$\|u_i\|_{C^{k,\alpha}(\Gamma^i_{+\delta_i})} \to \infty, \quad \text{as } i \to \infty. \tag{2.10}$$

We may assume that $\Omega^i \in \mathcal{M}_0$ as we can move it into $\bar{\mathbf{B}}^m(0, d_0)$ by a translation. By Proposition 2.3 there exist a subsequence $\{\Omega^l\}$ of $\{\Omega^i\}$, a domain $\Omega \in \mathcal{M}_0$ and $C^{1,\alpha}$ diffeomorphisms $\mathbf{f}_l : \bar{\Omega} \to \bar{\Omega}^l$ so that $\mathbf{f}_l \to \mathbf{I}_\Omega$ in $C^{1,\alpha/2}$. Setting $g_l = \mathbf{f}_l{}^* g_0$, we can identify (Ω_l, g_0) with (Ω, g_l). Clearly $g_l \to g_0$ in $C^{\alpha/2}$. Denote $u_l = u_l \circ \mathbf{f}_l$, $\varphi_l = \varphi_l \circ \mathbf{f}_l$. By Lemma 2.5, we have a subsequence $\{u_j\}$ of $\{u_l\}$ strongly converging to an energy minimizer $u \in \mathcal{U}_\varphi$, where $\varphi \in C^{k,\alpha}(\partial\Omega, N)$ and $\varphi = \lim \varphi_j$ in $C^{k,\alpha/2}$. By the boundary regularity theorem in [**SU2**], $u \in C^{k,\alpha}(\Gamma_{+\delta_4}, N)$ for some $\delta_4 > 0$ $(\Gamma = \partial\Omega)$.

We now show that $\sup_j \|u_j\|_{C^{k,\alpha}(\Gamma^j_{+\delta_5})} < \infty$ for some $\delta_5 > 0$. This will contradict to (2.10). In fact, we only need to show that

$$\sup_j \|u_j\|_{C^1(\Gamma^j_{+2\delta_5})} < \infty, \tag{2.11}$$

as once (2.11) is known, by applying the linear elliptic theory to the harmonic map equation that u_j satisfies, $\|u_j\|_{C^{k,\alpha}(\Gamma^j_{+\delta_5})}$ can be estimated in terms of the right hand of (2.11), $\|\varphi_j\|_{C^{2,\alpha}(\partial\Omega^j)} \le K$, d_0, δ_0, Λ_0, δ_5 and N, but independent of j.

We use Theorem 3.1 in [**SU1**] and Theorem 2.2 in [**SR**] (or Theorem 2.6[**SR**] and regularity estimate 1.5 [**SU2**] for boundary case). Let $\varepsilon > 0$ be a positive number less than ε, $\bar{\varepsilon}$, the constants in these Theorems. Since $u \in C^{k,\alpha}(\Gamma_{+\delta_4})$, there exists an $r \in (0, \delta_4/8]$, such that

$$r^{2-m} \int_{B(z,3r)} |\nabla u|^2 dx \leq \frac{\varepsilon}{2}, \tag{2.12}$$

for any $z \in \Gamma_{+4r}$. Where $B(z, 3r) \equiv \mathbf{B}(z, 3r) \cap \Gamma_{+4r}$. Take finite points $x_1, \ldots, x_p \in \Gamma_{+4r}$ so that $\Gamma_{+4r} \subset \cup_{\nu=1}^{p} \mathbf{B}(x_\nu, r)$.

Since $\mathbf{f}_j \to \mathbf{I}$ in $C^{1,\alpha/2}$, $u_j \to u$ in $L^{1,2}(\Omega, N)$, $g_i \to g_0$ in $C^{\alpha/2}$ and (2.12) holds for each x_ν, we have the followings (for j sufficiently large):

$$\begin{aligned} &\|\mathbf{f}_j - \mathbf{I}\|_{C^1(\Omega)} \leq 2; \\ &\|\mathbf{f}_j^{-1} - \mathbf{I}_{\Omega^j}\|_{C^1(\Omega^j)} \leq 2; \\ &r^{2-m} E\big(u_j, g_j, B(x_\nu, 3r)\big) \leq \varepsilon, \end{aligned} \tag{2.13}$$

where $E\big(u_j, g_j, B(x_\nu, 3r)\big)$ is the energy of u_j on $B(x_\nu, 3r)$ with respect to metric g_j. Also we have $\mathbf{f}_j^{-1}(\Gamma_{+r}^j) \subset \Gamma_{+2r}$.

Now for any $y \in \Gamma_{+r}^j$, there is some $1 \leq \nu \leq p$ so that $\mathbf{f}_j^{-1}(y) \in \mathbf{B}(x_\nu, r)$, therefore, $\mathbf{f}_j^{-1}(B(y, r)) \subset \mathbf{B}(x_\nu, 3r)$, where $B(y, r) \equiv \mathbf{B}(y, r) \cap \Gamma_{+r}^j$. From (2.13)

$$\begin{aligned} &r^{2-m} E\big(u_j, g_0, B(y, r)\big) \\ &= r^{2-m} E\big(u_j, g_j, \mathbf{f}_j^{-1}(B(y, r))\big) \\ &\leq r^{2-m} E\big(u_j, g_j, B(x_\nu, 3r)\big) \leq \varepsilon. \end{aligned}$$

So the small energy conditions in Theorem 3.1 in [**SU1**] and Theorem 2.2 in [**SR**] (or Theorem 2.6[**SR**] and regularity estimate 1.5 [**SU2**], respectively) hold, therefore u_j is α-Hölder continuous and then is $C^{k,\alpha}$ on Γ_{+2r}^j, and

$$\sup_{B(y,r/2)} |\nabla u_j| \leq C\big[r^{-2}\varepsilon + K\big], \quad \text{for any} \quad y \in \Gamma_{+r}^j,$$

where C depends only on m and N (and the curvature of $\partial\Gamma_{+r}^j$, which is bounded in terms of Λ_0 for boundary case). This implies (2.11) with $\delta_5 = r/2$.

Thus we finish the proof of Theorem 2.1. ■

From the proof of Theorem 2.1, it is obvious that Theorem 2.1 holds for a subfamily $\mathcal{M}_1$ of $\mathcal{M}$ which is closed under the convergence described in Proposition 2.3. In particular, we have the following corollaries, the second being needed in §4.

First, suppose that Ω is a given bounded $C^{2,\alpha}$ domain. Letting $\mathcal{M}_1 = \{\Omega\}$, we then have

Corollary 2.7. *Let Ω and N be the same as in Lemma 2.4. For any constant $K > 0$, there are positive constants δ and C depending only on Ω, K and N so that any energy minimizer $u \in \mathcal{U}_\varphi$ with $\varphi \in C^{k,\alpha}(\partial\Omega, N)$, $\|\varphi\|_{C^{k,\alpha}} \le K$ is $C^{2,\alpha}$ on $\Gamma_{+\delta}$ and*

$$\|u\|_{C^{2,\alpha}(\Gamma_{+\delta})} \le C.$$

Furthermore, from Corollary 2.7, one sees (from the the proof of Lemma 2.4) that the constant C in Lemma 2.4 actually depends only on Ω, N, $\|\varphi\|_{C^{1,\alpha}}$ and $\|\psi\|_{C^{1,\alpha}}$. It turns out that we have the following

Corollary 2.8. (Stability of Energy). *Suppose that Ω, N are the same as in Lemma 2.4. Then for $K > 0$, there exists a positive constant C depending only on K, Ω and N so that if φ, $\psi \in C^{1,\alpha}$ with $\|\varphi\|_{C^{1,\alpha}}$, $\|\psi\|_{C^{1,\alpha}} \le K$ and $u \in \mathcal{U}_\varphi$, $v \in \mathcal{U}_\psi$ are energy minimizers, then*

$$\left| \int_\Omega |\nabla u|^2 - \int_\Omega |\nabla v|^2 \right| \le C\|\varphi - \psi\|_{C^0(\partial\Omega,N)}.$$

Second, suppose that Ω is a bounded $C^{3,\alpha}$ domain. Then there exist positive numbers σ, d_0, δ_0 and Λ_0 so that $\Omega_r \in \mathcal{M}(d_0, \delta_0, \Lambda_0)$ for $|r| \le \sigma$. To check this, one notices that $\partial\Omega_r = \{\omega + r\mathbf{n}(\omega) : \omega \in \partial\Omega\}$, which is $C^{2,\alpha}$ diffeomorphic to $\partial\Omega$ when $|r|$ is small (see the Remark to Proposition 2.2). Now letting $\mathcal{M}_1 = \{\Omega_r : |r| \le \sigma\}$, we have

Corollary 2.9. *For any bounded $C^{3,\alpha}$ domain Ω, N as before, and constant $K > 0$, there exist positive constants σ, $\delta(\le \sigma)$ and C depending only on Ω, K and N so that if $r \in [-\sigma, \sigma]$, $\varphi \in C^{2,\alpha}$ with $\|\varphi\|_{C^{1,\alpha}} \le K$ and $u \in \mathcal{U}_{\varphi,\Omega_r}$ is an energy minimizer, then u is $C^{2,\alpha}$ in $\Omega_r \cap \Gamma_\delta$ and*

$$\|u\|_{C^{2,\alpha}(\Omega_r \cap \Gamma_\delta)} \le C.$$

3. A Uniqueness Result for Harmonic Map Equation

Here we prove a unique continuation result for harmonic maps, which may not be smooth. This result gives an alternative formulation of the nonuniqueness (3.5). For classical solutions of general elliptic equation or system, similar theorems were proved before, e.g., [**AN**][**MF2**]. In fact, the proof here shows that even weak solutions of general elliptic systems (linear or nonlinear) also enjoy this unique continuation property (e.g., the Jacobi fields of a harmonic map [**HM**]).

Lemma 3.1. *Let $\Omega \subseteq \mathbf{R}^m$ be a connected bounded $C^{2,\alpha}$ domain, N be C^3 compact Riemannian submanifold of $\mathbf{R}^n$ without boundary and $\varphi \in C^{2,\alpha}(\partial\Omega, N)$. If u and v are energy minimizers in $\mathcal{U}_\varphi$ and $\partial u/\partial \mathbf{n} = \partial v/\partial \mathbf{n}$ on a portion (nonempty and open) Σ of $\partial\Omega$, then $u \equiv v$ on Ω.*

Proof. By the boundary regularity theorem in [**SU2**], one knows that u and v are $C^{2,\alpha}$ in a neighborhood of $\partial\Omega$ and $\mathcal{H}^{m-2}(\mathcal{S}) = 0$, where $\mathcal{S}$ is the union of the singular sets of u and v. Let $\mathcal{R} = \Omega \setminus \mathcal{S}$, then $\mathcal{R}$ is open, as $\mathcal{S}$ is closed. (here *open* and *closed* are relative to $\bar{\Omega}$) Let

$$\mathcal{R}_1 = \{x \in \mathcal{R} : x \text{ has a neighborhood in which } u \equiv v\}.$$

Then clearly $\mathcal{R}_1$ is open. To prove that $\mathcal{R}_1 = \mathcal{R}$, it suffices to prove that $\mathcal{R}$ is connected, $\mathcal{R}_1$ is nonempty and closed. We proceed as follows.

$\mathcal{R}$ **is connected**. That $\mathcal{R}$ is path connected is well-known by the following argument. Let a and b are any points in $\mathcal{R}$. Since Ω is open and connected, and in which $\mathcal{R}$ in dense, one may connect a and b by a piecewise segment curve in Ω with vertices in $\mathcal{R}$. So it does not lose generality to assume that the segment $\overline{ab} \subset \Omega$ and that $a = 0$. Again since Ω is open, there is a positive number r_0 such that $\mathbf{B}^m(0, r_0)$ and $\mathbf{B}^m(b, r_0)$ are contained in $\mathcal{R}$, while the cone $C = \{\lambda x : 0 \le \lambda \le 1, x \in \mathbf{B}^m(b, r_0)\}$ is contained in Ω.

Now we consider the projectional map $\pi : \mathbf{R}^m \setminus \mathbf{B}^m(0, r_0) \to \mathbf{S}^{m-1}(r_0)$, $\pi(x) = r_0 \dfrac{x}{|x|}$. From that π is Lipchitz on $\mathbf{R}^m \setminus \mathbf{B}^m(0, r_0)$, which contains $\mathcal{S}$, and that $\mathcal{H}^{m-2}(\mathcal{S}) = 0$, one has $\mathcal{H}^{m-2}(\pi(\mathcal{S})) = 0$; therefore there exists a point $x_0 \notin \pi(\mathcal{S})$. This means that $\mathcal{R}$ contains the segment $\{\lambda x_0 : \lambda \in \mathbf{R}\} \cap C$. Take a point $c \in \mathbf{B}^m(b, r_0) \cap \{\lambda x_0 : \lambda \in \mathbf{R}\}$. Then $\overline{ac} \cup \overline{cb}$ is a path contained in $\mathcal{R}_1$ and connecting a and b.

$\mathcal{R}_1$ **is nonempty**. First we note that both u and v satisfy the harmonic map equation on $\mathcal{R}$ classically:

$$\mathcal{L}u \equiv \triangle u \,-\, A(u)(du, du) = 0, \qquad \mathcal{L}v = 0, \tag{3.1}$$

where $A(u)$ is the second fundmental form of N at u. A is of C^1 since N is of C^3.

Consider $w = u - v$. From (3.1) we have

$$\triangle w - \big(A(u) - A(v)\big)(du, du) - A(v)(du + dv, dw) = 0. \tag{3.2}$$

Take a point $\omega_0 \in \Sigma$ and a small positive number σ such that $\bar{\mathbf{B}} \cap \partial\Omega \subset \Sigma$ and that u, v are $C^{2,\alpha}$ on $\bar{\mathbf{B}} \cap \bar{\Omega}$, where $\mathbf{B} = \mathbf{B}^m(\omega_0, \sigma)$. Then (3.2) implies that w satisfies

$$|\triangle w| \le M\big(|w| + |\nabla w|\big), \tag{3.3}$$

on $\bar{\mathbf{B}} \cap \bar{\Omega}$, where the constant M depends on $\|u\|_{C^1}, \|v\|_{C^1}$ (on $\bar{\mathbf{B}} \cap \bar{\Omega}$) and $\|A\|_{C^1(N)}$. Extend w to $\bar{\mathbf{B}}$ such that $w(x) = 0$ for $x \in \bar{\mathbf{B}} \setminus \bar{\Omega}$, then $w \in C^2(\bar{\mathbf{B}})$ and (3.3) holds on $\mathbf{B}$ (note that all $\dfrac{\partial^k w}{\partial \mathbf{n}^k}$ and $\nabla_\omega^k w$ vanish on $\Sigma \cap \mathbf{B}$ for $k = 0, 1, 2$). By the Lemma 7.2 in [**MF2**], we have that $w \equiv 0$ on $\mathbf{B}$. So $\mathbf{B} \cap \Omega \subset \mathcal{R}_1$ and $\mathcal{R}_1$ is nonempty.

$\mathcal{R}_1$ **is closed**. Suppose $x_i \in \mathcal{R}_1$, and $x_i \to x \in \mathcal{R}$. Take a $\sigma > 0$ such that $\bar{\mathbf{B}}(x, \sigma) \subset \mathcal{R}$, then w satisfies (3.3) on $\bar{\mathbf{B}}(x, \sigma)$ with M depending on $\|u\|_{C^1}, \|v\|_{C^1}$ (on $\bar{\mathbf{B}}(x, \sigma$)) and $\|A\|_{C^1(N)}$. For i large we have that $x_i \in \mathbf{B}(x, \sigma/2)$; while $x_i \in \mathcal{R}_1$ implies that w is zero of infinite order at x_i, i.e., $\int_{|x - x_i| < \varepsilon} |\nabla u|^2 \leq O(\varepsilon^k)$ for any positive integer k and small ε. Therefore a unique continuation theorem [**AN**, Remark 3] implies, that $w \equiv 0$ on $\mathbf{B}(x, \sigma)$. So $x \in \mathcal{R}_1$ and then $\mathcal{R}_1$ is closed.

Lemma 3.1 is thus proved. ■

Checking the proof of Lemma 3.1, one sees a general unique continuation result is already shown. Consider the following quasilinear system:

$$
\begin{gathered}
Qu = a_{ij}(x, u, Du) D_{ij} u^\beta + b^\beta(x, u, Du), \\
x \in \Omega \subset \mathbf{R}^m, \quad u : \Omega \to \mathbf{R}^n, \quad a_{ij} = a_{ji}, \quad \beta = 1, ..., n.
\end{gathered}
\tag{3.4}
$$

Suppose that on $\bar{\Omega} \times \mathbf{R}^n \times \mathbf{R}^{m \times n}$ both $A = (a_{ij})_{m \times m}$ and $B = (b^\beta)_{1 \times n}$ are smooth, and furthermore, Q is elliptic, i.e., A is positive. Then we have

Proposition 3.2. *Let $\Omega \subset \mathbf{R}^m$ be a connected bounded $C^{2,\alpha}$ domain. Suppose that u, $v \in L^{1,2}(\Omega, N)$ are solutions of* (3.4) *satisfying the following*

i. $u = v$ on $\partial\Omega$, and $\partial u / \partial \mathbf{n} = \partial v / \partial \mathbf{n}$ on a portion of $\partial\Omega$;

ii. u and v are $C^{2,\alpha}$ on $\bar{\Omega} \setminus \mathcal{S}$, where $\mathcal{S}$ is a compact subset of Ω with $\mathcal{H}^{m-1}(\mathcal{S}) = 0$. Then $u \equiv v$ on Ω.

What we need in the application of §4 is the following fact implied by Lemma 3.1: If $u, v \in \mathcal{U}_\varphi$ are energy minimizers, as in Lemma 3.1, then

$$
u \neq v \ \text{ if and only if } \int_{\partial\Omega} \left| \frac{\partial u}{\partial \mathbf{n}} - \frac{\partial v}{\partial \mathbf{n}} \right|^2 d\omega > 0.
\tag{3.5}
$$

To prove this, one uses Lemma 3.1 and the fact that $\dfrac{\partial u}{\partial \mathbf{n}}$ and $\dfrac{\partial v}{\partial \mathbf{n}}$ are continuous on $\partial\Omega$ (from the boundary regularity theorem in [**SU2**]). In general, Lemma 3.1 implies the following corollary.

Corollary 3.3. *Let Ω and N be the same as in the Lemma 3.1 and $K > 0$ be any constant. Then for any $\varepsilon > 0$, there exists a constant $\theta > 0$, depending only on Ω, N, K and ε, so that if $\varphi, \psi \in C^{2,\alpha}(\partial\Omega, N)$ with $\|\varphi\|_{C^{2,\alpha}}$, $\|\psi\|_{C^{2,\alpha}} \leq K$, $u \in \mathcal{U}_\varphi$ and $v \in \mathcal{U}_\psi$ are energy minimizers satisfying*

$$\|\varphi - \psi\|_{C^0} \leq \theta,$$
$$\int_{\partial\Omega} \left| \frac{\partial u}{\partial \mathbf{n}} - \frac{\partial v}{\partial \mathbf{n}} \right|^2 d\omega \leq \theta,$$

then

$$\int_\Omega |\nabla u - \nabla v|^2 dx \leq \varepsilon.$$

Proof. Suppose for the sake of contradiction that the conclusion not true, then there exist a $\theta_0 > 0$ and two sequences φ_i and ψ_i of boundary data, and two bounded (in $C^{2,\alpha}$) sequences $u_i \in \mathcal{U}_{\varphi_i}$ and $v_i \in \mathcal{U}_{\psi_i}$ of energy minimizers such that

$$\|\varphi_i - \psi_i\|_{C^0} \to 0,$$
$$\int_{\partial\Omega} \left| \frac{\partial u_i}{\partial \mathbf{n}} - \frac{\partial v_i}{\partial \mathbf{n}} \right|^2 d\omega \to 0, \tag{3.6}$$

but

$$\int_\Omega |\nabla u_i - \nabla v_i|^2 dx \geq \theta_0. \tag{3.7}$$

By Lemma 2.5 and the uniform boundary regularity Theorem 2.1, we have a subsequence $\{j\}$ of $\{i\}$ so that

i. u_j and v_j converge strongly to energy minimizers u and v respectively;
ii. φ_j and ψ_j converge to φ, $\psi \in C^{2,\alpha}(\partial\Omega, N)$ in C^2 respectively, $u = \varphi$, $v = \psi$ on $\partial\Omega$;
iii. $\dfrac{\partial u_j}{\partial \mathbf{n}}$ and $\dfrac{\partial v}{\partial \mathbf{n}}_j$ converge to $\dfrac{\partial u}{\partial \mathbf{n}}$ and $\dfrac{\partial v}{\partial \mathbf{n}}$ in $C^1(\partial\Omega)$ respectively.

Taking the limits in (3.6) as $j \to \infty$, we get that $\varphi = \psi$ and $\dfrac{\partial u}{\partial \mathbf{n}} = \dfrac{\partial v}{\partial \mathbf{n}}$. By Lemma 3.1 we have $u = v$. This contradicts to the limit of (3.7), which says that $u \neq v$. ■

4. A Quantitative Uniqueness Lemma

In this section, we use Theorem 2.1 and Lemma 3.1 to prove a quantitative Lemma 4.1, which is crucial to employing F. Morgan's density argument in [**MF1**][**MF2**] to prove our main result Theorem 5.2.

In the rest of this paper, we will assume that Ω is a bounded $C^{3,\alpha}$ domain in $\mathbf{R}^m$ and N is a $C^{4,\alpha}$ compact submanifold of $\mathbf{R}^n$ without boundary. Let us denote Z the nonuniqueness set:

$$Z = \{\varphi \in C^{2,\alpha}(\partial\Omega, N) : \mathcal{U}_\varphi \text{ contains two different energy minimizers}\}.$$

For $K, \varepsilon \in (0, \infty)$ and $\varphi \in C^{2,\alpha}(\partial\Omega, N)$, we define

$$\begin{aligned}
(4.0)\ &\mathcal{B}(\varphi, K) = \{\psi \in C^{2,\alpha}(\partial\Omega, N) : \|\varphi - \psi\|_{C^{2,\alpha}} < K\};\\
&\mathcal{B}'(\varphi, K) = \{\psi \in C^{2,\alpha}(\partial\Omega, N) : \|\varphi - \psi\|_{C^2} < K\};\\
&Z_K = Z \cap \mathcal{B}(0, K);\\
&Z_{K,\varepsilon} = \{\varphi \in Z_K : \exists \text{ energy minimizers } u, v \in \mathcal{U}_\varphi \text{ with } \int_{\partial\Omega} \left|\frac{\partial u}{\partial \mathbf{n}} - \frac{\partial v}{\partial \mathbf{n}}\right|^2 dx > \varepsilon\}.
\end{aligned}$$

Obviously, $Z = \cup_{K>0} Z_K$; furthermore, by (3.5)

$$Z_K = \cup_{\varepsilon>0} Z_{K,\varepsilon}. \tag{4.1}$$

For any fixed $K, \varepsilon \in (0, \infty)$, we prove the following quantitative lemma by constructing comparison maps.

Lemma 4.1. *There exist positive numbers $\beta \in (0, 1)$, $\gamma \geq 1$ and $t_0 > 0$, depending only on Ω, N, K and ε, such that for any $\varphi \in C^{2,\alpha}(\partial\Omega, N)$ with $\|\varphi\|_{C^{2,\alpha}} < K$ and $0 < t \leq t_0$, there is a $\psi \in \mathcal{B}(\varphi, \gamma t)$ with $\mathcal{B}'(\psi, \beta t) \cap Z_{K,\varepsilon} = \emptyset$.*

Before we start the proof, we need some preparations. Following the discussion before Corollary 2.9, one has positive number $K' \geq K + 1$, depending only on K, Ω and N, so that for any $\varphi \in C^{2,\alpha}(\partial\Omega, N)$ with $\|\varphi\|_{C^{2,\alpha}} < K$, if one defines

$$\bar{\varphi} = \bar{\varphi}_r : \partial\Omega_r \to N, \quad \bar{\varphi}(\omega + r\mathbf{n}(\omega)) = \varphi(\omega) \tag{4.2}$$

for $|r| \leq \sigma$, then

$$\|\bar{\varphi}\|_{C^{2,\alpha}(\partial\Omega_r)} \leq K'.$$

Applying Corollary 2.9 with K replaced by K', we get positive numbers δ ($\leq \sigma$) and C_1, depending only on Ω, N and K, so that if $|r| \leq \sigma$, $\varphi_r \in \mathcal{C}_{K',\partial\Omega_r}$ and $u_r \in \mathcal{U}_{\varphi_r,\Omega_r}$ is an energy minimizer, then

$$\|u\|_{C^{2,\alpha}(\Omega_r \cap \Gamma_\delta)} \leq C_1. \tag{4.3}$$

In particular, if $0 \le t \le \sigma$, $\varphi \in \mathcal{C}_{K,\partial\Omega}$, $\bar\varphi_{-t}$ is defined as (4.2) above, and $\bar u \in \mathcal{U}_{\bar\varphi_{-t},\Omega_{-t}}$ is an energy minimizer, then

$$\|\bar u\|_{C^{2,\alpha}(\Omega_{-t}\cap\Gamma_\delta)} \le C_1. \tag{4.4}$$

Since N is of $C^{4,\alpha}$, we have $\pi_N \in C^{3,\alpha}(N_\tau)$ for some $\tau > 0$ depending only on N; therefore there exists a constant C_2 depending only on N such that, if u is a map from Ω (or $\partial\Omega$) to N, and v is a $C^{2,\alpha}$ map from Ω (or $\partial\Omega$) to $\mathbf{R}^n$ with $\|v\|_{C^0} \le \tau$, then

$$\begin{aligned} |D\pi(u+v)|(x) &\le |P_{u(x)}| + |D\pi(u+v) - D\pi(u)|(x) \le 1 + C_2|v(x)|, \\ |\nabla\pi(u+v)|(x) &= |(D\pi)(u+v) \bullet \nabla(u+v)|(x) \\ &\le (1 + C_2|v(x)|)|\nabla(u+v)|(x); \\ \|\pi(u+v) - u\|_{C^{k,\alpha}} &\le C_2\|v\|_{C^{k,\alpha}}, \quad (0 \le \alpha \le 1;\ 0 \le k \le 2). \end{aligned} \tag{4.5}$$

(Where the constant C_2 in the last inequality may also depend on $\|u\|_{C^{k,\alpha}}$, but in our following applications, $\|u\|_{C^{k,\alpha}}$ will be bounded by a constant depending only on K, Ω and N.)

Now suppose $\varphi \in \mathcal{C}_K$. For $0 \le t \le \sigma$, we define $\varphi_{-t} \in C^{2,\alpha}(\partial\Omega_{-t}, N)$ as in (4.2). Take an energy minimizer $\bar u \in \mathcal{U}_{\varphi_{-t},\Omega_{-t}}$, which satisfies (4.4). Especially, if $0 \le t \le \delta$, then $\frac{\partial \bar u}{\partial \mathbf{n}}|_{\partial\Omega} \in C^{1,\alpha}(\partial\Omega, \mathbf{R}^n)$. The ψ of Lemma 4.1 will be taken as a suitable approximation of $\varphi + t\frac{\partial\bar u}{\partial\mathbf{n}}|_{\partial\Omega}$. (Note that $\bar u|_\Omega$ is the unique energy minimizer with respect to its own boundary value $\bar u|_{\partial\Omega}$[**AL**, Theorem 4.1], which is close to φ. This already implies that Z is of first category. Lemma 4.1 is a stronger and quantitative result).

The rest of this section is devoted to the proof of Lemma 4.1. Unless otherwise indicated, the following constants $C_3, C_4, \dots$ depend only on Ω, N and K.

Proof of Lemma 4.1. Let $0 < \beta < 1$, $\gamma \ge \max\{C_1, C_2, 2\kappa, 1\}$ and $0 < t_0 \le \beta$ be constants to be chosen later. Suppose that for each $t \in (0, t_0)$, $\psi = \psi_t$ is a map in $C^{2,\alpha}(\partial\Omega, N)$ satisfying the following

$$\begin{aligned} \max_{\omega\in\partial\Omega} |\psi - \varphi - t\frac{\partial\bar u}{\partial\mathbf{n}}| &\le \beta t, \\ \|\psi - \varphi\|_{C^{2,\alpha}(\partial\Omega)} &\le \gamma t. \end{aligned} \tag{4.6}$$

Suppose also that $\theta = \theta_t \in C^{2,\alpha}(\partial\Omega, N)$ satisfies $\|\theta\|_{C^{2,\alpha}(\partial\Omega)} < K$ and

$$\|\theta - \psi\|_{C^2(\partial\Omega)} \le \beta t. \tag{4.7}$$

Let $v \in \mathcal{U}_{\theta,\Omega}$ be an energy minimizer. By (4.3), we have

$$\|v\|_{C^{2,\alpha}(\Gamma_{+\delta})} \le C_1. \tag{4.8}$$

Let $\tilde\theta(\omega) = v(\omega + t\mathbf{n}(\omega))$, $\psi^0(\omega) = \bar u(\omega)$ for $\omega \in \partial\Omega$. We need the following estimates (4.9)–(4.17):

By (4.6) and then by (4.7), we have

$$\max_{\omega\in\partial\Omega}|\psi-\varphi| \le t\max_{\omega\in\partial\Omega}\left|\frac{\partial\bar u}{\partial\mathbf{n}}\right| + \beta t \le (C_1+\beta)t \le 2C_1 t;$$
$$\max_{\omega\in\partial\Omega}|\theta-\varphi| \le \max_{\omega\in\partial\Omega}|\theta-\psi| + \max_{\omega\in\partial\Omega}|\psi-\varphi| \le 3C_1 t. \tag{4.9}$$

By (4.8),(4.9) and (4.4), we have

$$\begin{aligned}\max_{\omega\in\partial\Omega}|\tilde\theta-\varphi| &\le \max_{\omega\in\partial\Omega}|v(\omega+t\mathbf{n}(\omega))-\theta(\omega)| + \max_{\omega\in\partial\Omega}|\theta(\omega)-\varphi(\omega)|\\ &\le \max_{\omega\in\partial\Omega}\int_0^t\left|\frac{\partial v}{\partial\mathbf{n}}(\omega+r\mathbf{n}(\omega))\right|dr + 3C_1 t\\ &\le C_1 t + 3C_1 t \le 4C_1 t.\end{aligned} \tag{4.10}$$

For $0\le s\le\beta t$, by (4.7), (4.6), then (4.4), we have

$$\begin{aligned}&\max_{\omega\in\partial\Omega}|\theta-\bar u(\omega+s\mathbf{n}(\omega))| \le \max_{\omega\in\partial\Omega}|\theta-\psi|\\ &\quad+\max_{\omega\in\partial\Omega}\left|\psi-\varphi-t\frac{\partial\bar u}{\partial\mathbf{n}}\right| + \max_{\omega\in\partial\Omega}\left|\bar u(\omega+s\mathbf{n}(\omega))-\bar u(\omega-t\mathbf{n}(\omega))-t\frac{\partial\bar u}{\partial\mathbf{n}}(\omega)\right|\\ &\le \beta t+\beta t+\max_{\omega\in\partial\Omega}\left|\int_{-t}^s\frac{\partial\bar u}{\partial\mathbf{n}}(\omega+r\mathbf{n}(\omega))dr - t\frac{\partial\bar u}{\partial\mathbf{n}}(\omega)\right|\\ &\le 2\beta t + C_1 s + \max_{\omega\in\partial\Omega}\int_{-t}^0\int_r^0\left|\frac{\partial^2\bar u}{\partial\mathbf{n}^2}(\omega+\tau\mathbf{n}(\omega))\right|d\tau dr\\ &\le 2\beta t + C_1\beta t + \frac{1}{2}C_1 t^2 \le 4C_1\beta t.\end{aligned} \tag{4.11}$$

By(4.6), definition of $\bar u$, and (4.4), we have

$$\begin{aligned}\max_{\omega\in\partial\Omega}|\psi-\psi^0| &\le \max_{\omega\in\partial\Omega}\left|\psi-\varphi-t\frac{\partial\bar u}{\partial\mathbf{n}}\right| + \max_{\omega\in\partial\Omega}\left|\bar u(\omega)-\bar u(\omega-t\mathbf{n}(\omega))-t\frac{\partial\bar u}{\partial\mathbf{n}}\right|\\ &\le \beta t + \max_{\omega\in\partial\Omega}\int_{-t}^0\int_r^0\left|\frac{\partial^2\bar u}{\partial\mathbf{n}^2}(\omega+\tau\mathbf{n}(\omega))\right|d\tau dr\\ &\le \beta t + \frac{1}{2}C_1 t \le 2C_1 t.\end{aligned} \tag{4.12}$$

Similarly, by definitons of $\tilde\theta$, $\bar u$; by (4.4), (4.8) and then (4.11), we have

$$\begin{aligned}&\max_{\omega\in\partial\Omega}\left|\tilde\theta(\omega)-\varphi(\omega)-t\left(\frac{\partial\bar u}{\partial\mathbf{n}}+\frac{\partial v}{\partial\mathbf{n}}\right)(\omega)\right|\\ &\le \max_{\omega\in\partial\Omega}\left|v(\omega+t\mathbf{n}(\omega))-\theta(\omega)-t\frac{\partial\bar u}{\partial\mathbf{n}}\right|\\ &+\max\left|\bar u(\omega-t\mathbf{n}(\omega))-\bar u(\omega)-t\frac{\partial\bar u}{\partial\mathbf{n}}(\omega)\right| + \max_{\omega\in\partial\Omega}|\bar u(\omega)-\theta(\omega)|\\ &\le \frac{1}{2}C_1 t^2 + \frac{1}{2}C_1 t^2 + 4C_1\beta t \le 5C_1\beta t;\end{aligned} \tag{4.13}$$

$$|\tilde{\theta}(\omega) - \varphi(\omega)|^2 \le t^2\left(\left|\frac{\partial \bar{u}}{\partial \mathbf{n}} + \frac{\partial v}{\partial \mathbf{n}}\right| + 3C_1\beta\right)^2 \le t^2\left(\left|\frac{\partial \bar{u}}{\partial \mathbf{n}} + \frac{\partial v}{\partial \mathbf{n}}\right|^2 + C_3\beta\right).$$

Where the last line is obtained from (4.4), (4.8) and the following simple formula:

$$|a|^2 - \beta(|a|^2 + 2|c|^2) \le |a + c\beta|^2 \le |a|^2 + \beta(|a|^2 + 2|c|^2), \tag{4.14}$$

for any vectors a, c, and $\beta \in (0,1)$.

By (4.8), (4.14); and by (4.6), (4.7) we have

$$\begin{aligned}\max_{\omega\in\partial\Omega} |\nabla_\omega \tilde{\theta} - \nabla_\omega \theta| &\le \max_{\omega\in\partial\Omega} |\nabla_\omega v(\omega + t\mathbf{n}(\omega)) - \nabla_\omega v(\omega)| \\ &\le \int_0^t \left|\nabla_\omega \frac{\partial v}{\partial \mathbf{n}}(\omega + r\mathbf{n}(\omega))\right| dr \le C_1 t;\end{aligned}$$

$$|\nabla_\omega \tilde{\theta}|^2 \le |\nabla_\omega \theta|^2 + C_4\beta, \quad \text{for any} \quad \omega \in \partial\Omega; \tag{4.15}$$

$$\begin{aligned}&\max_{\omega\in\partial\Omega} |\nabla_\omega \tilde{\theta} - \nabla_\omega \varphi| \\ &\le \max_{\omega\in\partial\Omega} |\nabla_\omega \tilde{\theta} - \nabla_\omega \theta| + \max_{\omega\in\partial\Omega} |\nabla_\omega \theta - \nabla_\omega \psi| + \max_{\omega\in\partial\Omega} |\nabla_\omega \psi - \nabla_\omega \varphi| \\ &\le C_1 t + \beta t + \gamma t \le 3\gamma t.\end{aligned}$$

For $|r| \le t$, $\omega \in \partial\Omega$, by (4.4); and then (4.14) we have

$$\left|\frac{\partial \bar{u}}{\partial \mathbf{n}}(\omega + r\mathbf{n}(\omega)) - \frac{\partial \bar{u}}{\partial \mathbf{n}}(\omega)\right| \le \int_0^r \left|\frac{\partial^2 \bar{u}}{\partial \mathbf{n}^2}(\omega + \mathbf{n}(\omega)s)\right| ds \le C_1 t; \tag{4.16}$$

$$\left|\frac{\partial \bar{u}}{\partial \mathbf{n}}(\omega + r\mathbf{n}(\omega))\right|^2 \ge \left|\frac{\partial \bar{u}}{\partial \mathbf{n}}(\omega)\right|^2 - 3C_1^2 t.$$

Similarly, we get

$$\begin{aligned}|\nabla_\omega \bar{u}(\omega + r\mathbf{n}(\omega))|^2 &\ge |\nabla_\omega \bar{\varphi}(\omega - t\mathbf{n}(\omega))|^2 - 3C_1^2 t \\ &= |\nabla_\omega \varphi(\omega)|^2 - 3C_1^2 t, \qquad |r| \le t; \\ \left|\frac{\partial v}{\partial \mathbf{n}}(\omega + r\mathbf{n}(\omega))\right|^2 &\ge \left|\frac{\partial v}{\partial \mathbf{n}}(\omega)\right|^2 - 3C_1^2 t, \qquad 0 \le r \le t; \\ |\nabla_\omega v(\omega + r\mathbf{n}(\omega))|^2 &\ge |\nabla_\omega \theta(\omega)|^2 - 3C_1^2 t, \qquad 0 \le r \le t.\end{aligned} \tag{4.17}$$

Now we define $p : \Gamma_t \to N_\tau$,

$$p(x) = p(\omega + r\mathbf{n}(\omega)) = (1 - \eta(r))\varphi(\omega) + \eta(r)\tilde{\theta}(\omega), \quad |r| \le t,$$

where $\eta(r)=(t+r)/2t$; furthermore, we define $P:\Omega_{-t}\to N$,

$$P(x)=\begin{cases}\pi(p(x)), & x\in\Gamma_t;\\ v(x), & x\in\Omega_t,\end{cases}$$

then $P\in\mathcal{U}_{\varphi_{-t},\Omega_{-t}}$. Also we define $q:\Gamma_{+\beta t}\to N_\tau$,

$$q(x)=q(\omega+r\mathbf{n}(\omega))=(1-\zeta(r))\theta(\omega)+\zeta(r)\bar{u}(\omega+\mathbf{n}(\omega)\beta t),\quad 0\le r\le\beta t,$$

where $\zeta(r)=r/\beta t$; and furthermore we define $Q:\Omega\to N$

$$Q(x)=\begin{cases}\pi(q(x)), & x\in\Gamma_{+\beta t};\\ \bar{u}(x), & x\in\Omega_{\beta t},\end{cases}$$

then $Q\in\mathcal{U}_{\theta,\Omega}$.

Notice that from (4.10) (4.11), one has that, when $t\le\frac{\tau}{4C_1}$, π and therefore P and Q are well-defined.

From the minimalities of $\bar{u}\in\mathcal{U}_{\varphi_{-t},\Omega_{-t}}$ and $v\in\mathcal{U}_{\theta,\Omega}$, we have

$$E(\bar{u},\Omega_{-t})+E(v,\Omega)\le E(P,\Omega_{-t})+E(Q,\Omega).$$

Cancelling the common parts on both sides, we get

$$\int_{\Gamma_{-t}\cup\Gamma_{+\beta t}}|\nabla\bar{u}|^2dx+\int_{\Gamma_{+t}}|\nabla v|^2dx\le\int_{\Gamma_t}|\nabla\pi(p)|^2dx+\int_{\Gamma_{+\beta t}}|\nabla\pi(q)|^2dx. \tag{4.18}$$

Now we estimate each term in (4.18), as follows in (4.18)–(4.22):

By (4.5), (4.9); and then (2.2) we have

$$\begin{aligned}\int_{\Gamma_t}|\nabla\pi(p)|^2dx&\le(1+3C_1C_2t)^2\int_{\Gamma_t}|\nabla p|^2dx\\ &\le(1+3C_1C_2t)^2(1+(2m+2)\kappa t)\int_{-t}^{t}\int_{\partial\Omega}\left(\left|\frac{\partial p}{\partial\mathbf{n}}\right|^2+|\nabla_\omega p|^2\right)d\omega dr;\end{aligned}$$

$$\left|\frac{\partial p}{\partial\mathbf{n}}\right|^2=\frac{1}{4t^2}|\varphi(\omega)-\tilde{\theta}(\omega)|^2\le\frac{1}{4}\left(\left|\frac{\partial\bar{u}}{\partial\mathbf{n}}+\frac{\partial v}{\partial\mathbf{n}}\right|^2+C_3\right);\qquad\text{by (4.13)}$$

$$\begin{aligned}|\nabla_\omega p|^2&=\left(\frac{1}{2}|\nabla_\omega\tilde{\theta}+\nabla_\omega\varphi|+|\frac{1}{2}-\eta||\nabla_\omega\tilde{\theta}-\nabla_\omega\varphi|\right)^2\\ &\le\left(\frac{1}{2}|\nabla_\omega\tilde{\theta}+\nabla_\omega\varphi|+3\gamma t\right)^2 &&\text{by (4.15)}\\ &\le\frac{1}{2}\left(|\nabla_\omega\tilde{\theta}|^2+|\nabla_\omega\varphi|^2\right)+t(C_1^2+18\gamma^2) &&\text{by (4.14)}\\ &\le\frac{1}{2}\left(|\nabla_\omega\theta|^2+|\nabla_\omega\varphi|^2\right)+C_5\beta+18\gamma^2t, &&\text{by (4.15)}\end{aligned}$$

where $C_5 = C_4 + C_1^2$. Therefore, we have

(4.19)

$$\begin{aligned}
&\int_{\Gamma_t} |\nabla \pi(p)|^2 dx \\
&\le (1 + C_6 t) \int_{-t}^{t} \int_{\partial\Omega} \left(\frac{1}{4} \left| \frac{\partial \bar{u}}{\partial \mathbf{n}} + \frac{\partial v}{\partial \mathbf{n}} \right|^2 + \frac{1}{2} \left(|\nabla_\omega \theta|^2 + |\nabla_\omega \varphi|^2 \right) + (C_5 + C_3)\beta + 18\gamma^2 t \right) d\omega dr \\
&\le t \int_{\partial\Omega} \left(\frac{1}{2} \left| \frac{\partial \bar{u}}{\partial \mathbf{n}} + \frac{\partial v}{\partial \mathbf{n}} \right|^2 + |\nabla_\omega \theta|^2 + |\nabla_\omega \varphi|^2 \right) d\omega + C_7 \beta t + C_7 \gamma^2 t^2
\end{aligned}$$

where we used the fact that the whole integral $\int_{\partial\Omega} \cdots$ (except that of the term $18\gamma^2 t$) is bounded, by (4.4), (4.8) and (4.15).

Again by (4.5), (4.11); and then (2.2), we have

$$\begin{aligned}
\int_{\Gamma_{+\beta t}} |\nabla \pi(q)|^2 dx &\le (1 + 4C_1 C_2 \beta t)^2 \int_{\Gamma_{+\beta t}} |\nabla\ q|^2 dx \\
&\le (1 + 4C_1 C_2 \beta t)^2 (1 + (2m+2)\kappa t) \int_0^{\beta t} \int_{\partial\Omega} \left(\left| \frac{\partial q}{\partial \mathbf{n}} \right|^2 + |\nabla_\omega q|^2 \right) d\omega dr;
\end{aligned}$$

$$\begin{aligned}
\left| \frac{\partial q}{\partial \mathbf{n}} \right|^2 &\le \max_{\omega \in \partial\Omega} \frac{1}{\beta t} |\theta(\omega) - \bar{u}(\omega + \beta t \mathbf{n}(\omega))|^2 \le 16 C_1^2; \\
|\nabla_\omega q|^2 &= |(1 - \zeta(r)) \nabla_\omega \theta(\omega) + \zeta(r) \nabla_\omega \bar{u}(\omega + \mathbf{n}(\omega)\beta t)|^2 \le C_8,
\end{aligned}$$

where the last inequality is from (4.4) and the fact that $\theta \in \mathcal{C}_K$; therefore we have

$$\int_{\Gamma_{+t}} |\nabla \pi(q)|^2 dx \le C_9 \beta t. \tag{4.20}$$

By (2.2), (4.16) and (4.17), we have

(4.21)

$$\begin{aligned}
\int_{\Gamma_{-t} \cup \Gamma_{+\beta t}} |\nabla \bar{u}|^2 dx &\ge (1 - (2m+2)\kappa t) \int_{-t}^{\beta t} \int_{\partial\Omega} \left(\left| \frac{\partial \bar{u}}{\partial \mathbf{n}} \right|^2 + |\nabla_\omega \bar{u}|^2 \right) (\omega, r) d\omega dr \\
&\ge (1 - (2m+2)\kappa t) \int_{-t}^{\beta t} \int_{\partial\Omega} \left(\left| \frac{\partial \bar{u}}{\partial \mathbf{n}} \right|^2 + |\nabla_\omega \varphi|^2 - 6C_1^2 t \right) d\omega dr \\
&\ge (1 - (2m+2)\kappa t)(1 + \beta) t \left(\int_{\partial\Omega} \left[\left| \frac{\partial \bar{u}}{\partial \mathbf{n}} \right|^2 + |\nabla_\omega \varphi|^2 \right] d\omega - 6C_1^2 \mathrm{area}(\partial\Omega) t^2 \right) \\
&\ge t \int_{\partial\Omega} \left(\left| \frac{\partial \bar{u}}{\partial \mathbf{n}} \right|^2 + |\nabla_\omega \varphi|^2 \right) d\omega - C_{10} \beta t,
\end{aligned}$$

where again we used the fact the whole integral $\int_{\partial\Omega}\cdots$ is bounded. Similarly,

$$\int_{\Gamma_{+t}} |\nabla v|^2 dx \geq t\int_{\partial\Omega}\left(\left|\frac{\partial v}{\partial \mathbf{n}}\right|^2 + |\nabla_\omega \theta|^2\right)d\omega - C_{11}\beta t. \tag{4.22}$$

Substituting (4.19)–(4.22) into (4.18), we get

$$\frac{1}{2}\int_{\partial\Omega}\left|\frac{\partial \bar{u}}{\partial \mathbf{n}} - \frac{\partial v}{\partial \mathbf{n}}\right|^2 d\omega \leq C_{12}\beta + C_7\gamma^2 t, \tag{4.23}$$

where $C_{12} = C_7 + C_9 + C_{10} + C_{11}$.

Now we start to choose β, γ, t_0 and determine ψ (depending on $0 < t \leq t_0$ and φ).

Take $\beta = \min\{\frac{\varepsilon}{4C_{12}}, 1\}$.

For $\varphi \in C^{2,\alpha}(\partial\Omega, N)$ and $t > 0$ small, let $\bar{u}$ be as in (4.4). Define $h : \Gamma_\delta \to \mathbf{R}^n$ by

$$h(x) = h(\omega + r\mathbf{n}(\omega)) = \frac{\partial \bar{u}}{\partial \mathbf{n}}(\pi(x)) = \frac{\partial \bar{u}}{\partial \mathbf{n}}(\omega) \quad \text{for } x \in \Gamma_\delta).$$

Then we define

$$\psi(\omega) = \pi_N[\varphi(\omega) + th * \xi_s(\omega)], \tag{4.24}$$

where $\xi \in C_0^\infty(\mathbf{B}^m(0,1))$ is a nonnegative modifier satisfying $\int \xi dx = 1$; $\xi_s(x) = s^{-m}\xi(\frac{x}{s})$, and $s \in (0, \delta)$ is to be chosen, and

$$h * \xi_s(x) = \int h(x - sz)\xi(z)dz = \int h(y)\xi_s(y - x)dy.$$

Clearly, $\|h * \xi_s\|_{C^0(\partial\Omega)} \leq C_1$. So ψ is well-defined when $0 < t \leq \min\{\tau/C_1, \delta\}$.

By Taylor's formula,

$$\begin{aligned}
\max_{\omega\in\partial\Omega}\left|\psi(\omega) - \varphi(\omega) - t\frac{\partial \bar{u}}{\partial \mathbf{n}}(\omega)\right| &= \max_{\omega\in\partial\Omega}\left|\pi_N\big(\varphi(\omega) + th * \xi_s(\omega)\big) - \varphi(\omega) - t\frac{\partial \bar{u}}{\partial \mathbf{n}}(\omega)\right| \\
&\leq t\max_{\omega\in\partial\Omega}\left|D\pi_N(\varphi(\omega))h * \xi_s(\omega) - \frac{\partial \bar{u}}{\partial \mathbf{n}}(\omega)\right| + \\
&\quad \left|\int_0^t (t - r)D^2\pi_N\big(\varphi(\omega) + rh * \xi_s(\omega)\big)\big(h * \xi_s(\omega), h * \xi_s(\omega)\big)dr\right| \\
&\leq t\left|\int_{\mathbf{R}^m} D\pi_N(\varphi(\omega))h(\omega - sz)\xi(z)dz - \frac{\partial \bar{u}}{\partial \mathbf{n}}(\omega)\right| + C_{13}t^2,
\end{aligned} \tag{4.25}$$

where C_{13} depends on $\|\pi_N\|_{C^2}$ and $C_1 \geq \|h * \xi_s\|_{C^1}$.

Note that $\varphi(\omega) = \bar{u}(\omega - t\mathbf{n}(\omega)) \in N$ and $\dfrac{\partial \bar{u}}{\partial \mathbf{n}}(\omega - t\mathbf{n}(\omega)) \in T_{\varphi(\omega)}(N)$. From (2.7) with $x = \varphi(\omega)$), one has that $D\pi_N(\varphi(\omega))$ is the identity on $T_{\varphi(\omega)}(N)$. So

$$D\pi_N(\varphi(\omega))\frac{\partial \bar{u}}{\partial \mathbf{n}}(\omega - t\mathbf{n}(\omega)) = \frac{\partial \bar{u}}{\partial \mathbf{n}}(\omega - t\mathbf{n}(\omega)). \tag{4.26}$$

On the other hand by (4.4), we have (for $|z| \le 1$)

$$\begin{aligned} \left| h(\omega - sz) - \frac{\partial \bar{u}}{\partial \mathbf{n}}(\omega - t\mathbf{n}(\omega)) \right| &= \left| \frac{\partial \bar{u}}{\partial \mathbf{n}}(\pi(\omega - sz)) - \frac{\partial \bar{u}}{\partial \mathbf{n}}(\omega - t\mathbf{n}(\omega) \right| \\ &\le C_1 |\pi(\omega - sz) - \omega + t\mathbf{n}(\omega)| \le C_1(C_2 s + t). \end{aligned} \tag{4.27}$$

So from (4.25)–(4.27), we have

$$\begin{aligned} &\max_{\omega \in \partial\Omega} \left| \psi(\omega) - \varphi(\omega) - t\frac{\partial \bar{u}}{\partial \mathbf{n}}(\omega) \right| \\ &\le t\left| \frac{\partial \bar{u}}{\partial \mathbf{n}}(\omega - t\mathbf{n}(\omega)) - \frac{\partial \bar{u}}{\partial \mathbf{n}}(\omega) \right| + C_1 \|\pi\|_{C^1} t(t + C_2 s) + C_{13} t^2 \\ &\le \frac{1}{2} C_1 t^2 + C_1 \|\pi\|_{C^1} t^2 + C_{13} t^2 + C_1 C_2 \|\pi\|_{C^1} ts \le C_{14} st + C_{15} t^2. \end{aligned} \tag{4.28}$$

Take s_0 =min$\{\frac{\beta}{2C_{14}}, \delta\}$, and suppose for a moment that $t_0 \le \frac{\beta}{2C_{15}}$. Let ψ be defined by (4.24) with this fixed modifier ξ_{s_0}, and $0 < t \le t_0$. Then (4.28) and (4.5) imply That ψ satisfies our pre-assumed condition (4.6):

$$\begin{aligned} \max_{\omega \in \partial\Omega} \left| \psi - \varphi - t\frac{\partial \bar{u}}{\partial \mathbf{n}} \right| &\le \beta t; \\ \|\psi - \varphi\|_{C^{2,\alpha}(\partial\Omega)} &= \|\pi(\varphi + th * \xi_{s_0}) - \varphi\|_{C^{2,\alpha}(\partial\Omega)} \\ &\le C_2 t \|h * \xi_{s_0}\|_{C^{2,\alpha}(\partial\Omega)} \\ &\le C_2 t \|h\|_{C^0} \|\xi_{s_0}\|_{C^{2,\alpha}} \le \gamma t, \end{aligned}$$

for $\gamma = C_1 C_2 \|\xi_{s_0}\|_{C^{2,\alpha}(\partial\Omega)}$.

Now take $t_0 = \min\{\frac{\beta}{2C_{15}}, \frac{\varepsilon}{8C_7\gamma^2}, \frac{\tau}{4C_1}, \delta, \sigma\}$, then when $0 < t \le t_0$, from (4.23) and the choices of β and t_0, we have

$$\int_{\partial\Omega} \left| \frac{\partial \bar{u}}{\partial \mathbf{n}} - \frac{\partial v}{\partial \mathbf{n}} \right|^2 d\omega \le \frac{\varepsilon}{4}.$$

Especially, if both $v, w \in \mathcal{U}_\theta$ are energy minimizers, then

$$\int_{\partial\Omega} \left| \frac{\partial w}{\partial \mathbf{n}} - \frac{\partial v}{\partial \mathbf{n}} \right|^2 d\omega \le \varepsilon,$$

i.e., $\theta \notin Z_{K,\varepsilon}$ (by definition of $Z_{K,\varepsilon}$). In summary, we have constants β, γ and t_0 defined as above, depending on Ω, N, K and ε only, so that for any $\varphi \in C^{2,\alpha}(\partial\Omega, N)$ with $\|\varphi\|_{C^{2,\alpha}} < K$, and $0 < t \le t_0$, there exists a $\psi \in \mathcal{B}(\varphi, \gamma t)$ (defined by (4.24)), satisfying that if $\theta \in \mathcal{B}'(\psi, \beta t)$, then $\theta \notin Z_{K,\varepsilon}$.

Thus we finish the proof of Lemma 4.1. ■

5. The Uniqueness Theorem

Now we prove our main result Theorem 5.1: $\mu(Z) = 0$ where μ is a measure to be constructed below.

In fact, we have to restrict ourselves to a smaller space $C^{2,\alpha+}(\partial\Omega, N)$, the $\| \ \|_{C^{2,\alpha}}$ closure of $C^3(\partial\Omega, N)$ in $C^{2,\alpha}(\partial\Omega, N)$, on which a measure μ will be constructed, and then $\mu(Z \cap C^{2,\alpha+}(\partial\Omega, N)) = 0$ is shown. We require that μ have positive value on each nonempty open subset; therefore, such a measure does not exist on $C^{2,\alpha}(\partial\Omega, N)$, because it is generally nonseparable. (See [**MF2**, 6.1], [**WB**, 1.5].)

Let $\tau > 0$ be a positive number such that for $x \in N_\tau = \{x \in \mathbf{R}^n : \operatorname{dist}(x, N) < \tau\}$, there exists unique nearest point $\pi(x)$. Denote

$$C^{2,\alpha+}(\partial\Omega, N_\tau) = \{\varphi \in C^{2,\alpha+}(\partial\Omega, \mathbf{R}^n) : \varphi(x) \in N_\tau \quad \text{for all } x \in \partial\Omega\}.$$

Clearly $C^{2,\alpha+}(\partial\Omega, N_\tau)$ is nonempty open subset of $C^{2,\alpha+}(\partial\Omega, \mathbf{R}^n)$. Now we consider the following extension Π of π:

$$\begin{aligned} &\Pi : C^{2,\alpha+}(\partial\Omega, N_\tau) \to C^{2,\alpha+}(\partial\Omega, N), \\ &\varphi \to \Pi\varphi, \quad (\Pi\varphi)(x) = \pi(\varphi(x)). \end{aligned}$$

Obviously Π is surjective. From (4.5), Π is also continuous.

Let μ_0 be the measure on $C^{2,\alpha+}(\partial\Omega, \mathbf{R}^n)$ defined by F. Morgan in [**MF2**], which is positive at its each nonempty open subset (and so is $\mu_0|_{C^{2,\alpha+}(\partial\Omega, N_\tau)}$). Let $\mu = \Pi_\sharp \mu_0$, i.e.,

$$\mu(A) = \mu_0(\Pi^{-1}(A)) \quad \text{for} \quad A \subseteq C^{2,\alpha+}(\partial\Omega, N).$$

If A is open and nonempty, then $\Pi^{-1}(A)$ is also open and nonempty, by continuity and surjectivity of Π. Therefore, $\mu(A) = \mu_0(\Pi^{-1}(A))$ is positive. So μ is a needed measure on $C^{2,\alpha+}(\partial\Omega, N)$.

We may, and will, for convenience, extend μ to $C^{2,\alpha}(\partial\Omega, N)$ by

$$\mu(A) = \mu(A \cap C^{2,\alpha+}(\partial\Omega, N)), \quad A \subset C^{2,\alpha}(\partial\Omega, N).$$

(But $\mu(A)$ may be 0 even if A is open and nonempty in $C^{2,\alpha}(\partial\Omega, N)$.)

Now we are going to prove $\mu(Z) = 0$, i.e., $\mu_0(\Pi^{-1}(Z)) = 0$. By (4.1), it suffices to show that $\mu_0(\Pi^{-1}(Z_{K,\varepsilon})) = 0$ for all $K, \varepsilon \in (0, \infty)$. The quantitative Lemma 4.1, which was for the set $Z_{K,\varepsilon}$, can now be transferred to $\Pi^{-1}(Z_{K,\varepsilon})$, as follows. We use $\mathcal{B}_1$ and $\mathcal{B}_1'$ denote balls in $C^{2,\alpha}(\partial\Omega, N_\tau)$ (see (4.0) for definitions).

Lemma 5.1. *There exist positive numbers β_1, γ_1 and $t_1 = t_0$, depending only on Ω, N, K and ε, so that for any $\varphi_1 \in \Pi^{-1}(Z_{K,\varepsilon})$ and $0 < t \le t_1$, there is a $\psi_1 \in \mathcal{B}_1(\varphi_1, \gamma_1 t)$ with $\mathcal{B}_1'(\psi_1, \beta_1 t) \cap \Pi^{-1}(Z_{K,\varepsilon}) = \emptyset$.*

Proof. Suppose $\varphi_1 \in \Pi^{-1}(Z_{K,\varepsilon})$. Then $\Pi\varphi_1 \in Z_{K,\varepsilon}$. Let $\psi \in \mathcal{B}(\Pi\varphi_1, \gamma t)$ be the existed ψ in Lemma 4.1 corresponding to $\Pi\varphi_1$ and $0 < t \le t_0$. Take

$$\psi_1(x) = \psi(x) + P^\perp_{\psi(x)}(\varphi_1(x) - (\Pi\varphi_1)(x)), \quad x \in \partial\Omega.$$

Where $P^\perp_y : \mathbf{R}^n \to T^\perp_y(N)$ $(y \in N)$ is the orthogonal projection. Note that N is $C^{4,\alpha}$ implies that $P^\perp_y$ is $C^{3,\alpha}$ in y, and $\varphi_1 \in C^{2,\alpha}(\partial\Omega, N_\tau)$. Furthermore, from

$$\psi_1(x) - \varphi_1(x) = \psi(x) - (\Pi\varphi_1)(x) + [P^\perp_{\psi_1(x)} - P^\perp_{(\Pi\varphi_1)(x)}](\varphi(x) - (\Pi\varphi_1)(x)),$$

and $\psi \in \mathcal{B}(\Pi\varphi_1, \gamma t)$, one has that, for a constant γ_1 which depends only on Ω, N and K,

$$\|\psi_1 - \varphi_1\|_{C^{2,\alpha}(\partial\Omega)} \le \gamma_1 t. \tag{5.1}$$

Take a $\beta_1 < \beta/C_2$ and $t_1 = t_0$. Now if $\|\theta_1 - \psi_1\|_{C^2(\partial\Omega)} \le \beta_1 t$, then by (4.5),

$$\|\Pi\theta_1 - \psi\|_{C^2(\partial\Omega)} = \|\Pi\theta_1 - \Pi\psi_1\|_{C^2(\partial\Omega)} \le C_2\beta_1 t < \beta t.$$

Therefore by the choices of ψ, we have that $\Pi\theta_1 \notin Z_{K,\varepsilon}$ or $\theta_1 \notin \Pi^{-1}(Z_{K,\varepsilon})$. Thus Lemma 5.1 is proved. ∎

The rest of the proof that $\mu_0(\Pi^{-1}(Z_{K,\varepsilon})) = 0$ from Lemma 5.1 follows exactly the density argument of [**MF1**, 7.8]. Here the Approximation Lemmas [**MF1** 4.6, **MF2** 4.4] are applied to

$$H = \{\frac{\psi_1 - \varphi_1}{t} : \varphi_1 \in \Pi^{-1}(Z_{K,\varepsilon}),\ 0 < t \le t_1,\ \text{and } \psi_1 \text{ as in Lemma 5.1}\}$$

with $\|\ \| = \|\ \|_{C^2(\partial\Omega)}$. Notice that, by (5.1), $H \subset C^{2,\alpha+}(\partial\Omega, \mathbf{R}^n)$ and $\|\frac{\psi_1-\varphi_1}{t}\|_{C^{2,\alpha}(\partial\Omega)} \le \gamma_1$. So H is equicontinuous up to second derivatives. (See [**MF1**, p271] and [**MF2**, p345] for details.)

Summarizing the discussions above, we have

Theorem 5.2. *For μ–almost all $\varphi \in C^{2,\alpha+}(\partial\Omega, N)$, there exists at most one minimizer of $\int_\Omega |\nabla u|^2$ with trace φ.*

§6. A Remark on the Domain

Finally we remark that Theorem 5.1 holds for the cases where $\Omega = M$ is a general Riemannian manifold.

Suppose that M^n is compact and smooth Riemannian submanifold of $\mathbf{R}^m$ with ∂M being $C^{3,\alpha}$. By extending M along its boundary in its outward normal direction, we may assume that M is contained in a bigger manifold $\tilde{M}$, which is also smooth and satisfies that $\operatorname{dist}(M, \partial\tilde{M}) \geq \sigma$ for some positive number σ.

Now consider the family $\mathcal{M} = \{M_r : |r| \leq \sigma\}$, where

$$\begin{aligned} M_r =& \{x \in \tilde{M} : \operatorname{dist}(x, M) < -r\}, \quad \text{if } r < 0; \\ M_r =& \{x \in M : \operatorname{dist}(x, \partial M) > r\}, \quad \text{if } r \geq 0. \end{aligned}$$

If σ is small, then each $M_r \in \mathcal{M}$ is smooth, with ∂M_r being $C^{2,\alpha}$. Obviously, $\mathcal{M}$ is compact in the sense of Proposition 2.3. Therefore, one can repeat the proof of Theorem 2.1 to show that the corresponding conclusion of Corollary 2.9 with Ω replaced by M still holds, which is the (essentially) only estimate we need to prove a corresponding Lemma 4.1. The rest of the discussions are essentially the same.

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