Strong Convergence of p-Harmonic Mappings

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Let Ω be a smooth bounded domain in $\mathbf{R}^n$, N be a compact smooth submanifold of $\mathbf{R}^k$, and $p \geq 2$. Recall [HL] that a *p-harmonic map* to N is a map $u \in W^{1,p}(\Omega, N)$ that is a weak solution of an equation of the form

$$\operatorname{div}\left(|\nabla u|^{p-2}\nabla u\right) \ + \ f(u, \nabla u) \ = \ 0 \ ,$$

where $|f(u, \nabla u)| \leq c_N |\nabla u|^p$, that is,

$$\int_\Omega \left(|\nabla u|^{p-2}\nabla u \cdot \nabla\zeta \ - f(u, \nabla u)\cdot\zeta\right)\, dx \ = \ 0.$$

for all $\zeta \in \mathcal{C}_0^\infty(\Omega, \mathbf{R}^k)$. The above equation then also holds for $\zeta \in W_0^{1,p}(\Omega, \mathbf{R}^k) \cap L^\infty$.

Here we show how $W^{1,p}$ weakly convergent sequences of p-harmonic maps are strongly convergent in $W^{1,q}$ for $1 < q < p < \infty$. First we prove some useful inequalities.

Lemma 1. *If $p \geq 2$ and $0 \leq \mu \leq \lambda$, then for all $a \geq 0$,*

$$(\lambda - \mu)^{p-1} \ \leq \ 2[(a + \lambda^2)^{\frac{p-2}{2}}\lambda - (a + \mu^2)^{\frac{p-2}{2}}\mu] \ .$$

Proof : Let $f_a(\lambda) = (a + \lambda^2)^{\frac{p-2}{2}}\lambda$.

In case $\frac{1}{2}\lambda \leq \mu \leq \lambda$ and $\mu < c < \lambda$,

$$(f_a)'(c) \ = \ (a + c^2)^{\frac{p-2}{2}} + (p-2)(a + c^2)^{\frac{p-4}{2}}c^2 \ \geq \ c^{p-2} + 0 \ \geq \ (\lambda - \mu)^{p-2}$$

so that the mean value theorem gives

$$f_a(\lambda) - f_a(\mu) \ \geq \ (\lambda - \mu)^{p-2}(\lambda - \mu) \ = \ (\lambda - \mu)^{p-1} \ .$$

In case $-\lambda \leq \mu < \frac{1}{2}\lambda$,

$$\begin{aligned}(\lambda - \mu)^{p-1} \ &\leq \ \lambda^{p-1} \ \leq \ (a + \lambda^2)^{\frac{p-2}{2}}\lambda \\ &\leq \ 2(a + \lambda^2)^{\frac{p-2}{2}}(\lambda - \mu) \leq \ 2\left[f_a(\lambda) - f_a(\mu)\right] \ .\end{aligned}$$

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Corollary 1. *If* $p \geq 2$*, then,*

$$\left(|y|^{p-2}y - |z|^{p-2}z\right) \cdot (y-z) \;\geq\; \frac{1}{2}|y-z|^p$$

for any vectors y *and* z *in* $\mathbf{R}^k$*.*

Proof : Switching y and z if necessary, we may write

$$y = x + \lambda w \ , \ z = x + \mu w \ ,$$

for some vectors x and w in $\mathbf{R}^k$ and numbers μ and λ with $|\mu| \leq \lambda,\ x \cdot w = 0$, and $|w| = 1$. Then

$$\begin{gathered} y - z = (\lambda - \mu)w \ , \\ y \cdot (y-z) = \lambda(\lambda-\mu) \ , \quad z \cdot (y-z) = \mu(\lambda-\mu) \ , \\ |y|^2 = |x|^2 + \lambda^2 \ , \quad |z|^2 = |x|^2 + \mu^2 \ . \end{gathered}$$

Thus, by Lemma 1,

$$\begin{aligned} |y-z|^p \ = (\lambda-\mu)^p \ &\leq \ 2^p[(|x|^2+\lambda^2)^{\frac{p-2}{2}}\lambda - (|x|^2+\mu^2)^{\frac{p-2}{2}}\mu](\lambda-\mu) \\ &= \ 2^p \left(|y|^{p-2}y - |z|^{p-2}z\right) \cdot (y-z) \ . \end{aligned}$$

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Remark. Such an inequality fails for $p < 2$ and $n \geq 2$ as is seen by taking $y_i = (-1, i)$ and $z_i = (1, i)$. However, Corollary 2 below gives a suitable integral inequality for $1 < p \leq 2$.

Lemma 2. *If* $1 < p \leq 2$*, then,*

$$\left(|y|^{p-2}y - |z|^{p-2}z\right) \cdot (y-z) \;\geq\; (p-1)\left(|y| + |z|\right)^{p-2}|y-z|^2$$

for any vectors y *and* z *in* $\mathbf{R}^k$*.*

Proof : Let $F(x) = |x|^{p-2}x$ so that

$$\frac{\partial F^i}{\partial x_j}(x) \ = \ |x|^{p-2}\delta_{ij} \ + \ (p-2)|x|^{p-4}x_i x_j \ .$$

Then, letting $z_t = z + t(y-z)$, we see from Schwarz's inequality that

$$
\begin{aligned}
(F(y)-F(z))\cdot(y-z) \;&=\; \int_0^1 \frac{d}{dt}F(z_t)\cdot(y-z)\,dt \\
&= \int_0^1 \sum_{i,j}\frac{\partial F^i}{\partial x_j}(z_t)(y_i-z_i)(y_j-z_j)\,dt \\
&= \int_0^1 |z_t|^{p-2}|y-z|^2\,dt \;+\; (p-2)\int_0^1 \sum_{i,j}|z_t|^{p-4}(z_t)_i(z_t)_j(y_i-z_i)(y_j-z_j)\,dt \\
&\geq \int_0^1 |z_t|^{p-2}|y-z|^2\,dt \;-\; (p-2)\int_0^1 |z_t|^{p-4}|z_t|^2|y-z|^2\,dt \\
&\geq (p-1)\left(|y|+|z|\right)^{p-2}|y-z|^2 \;.
\end{aligned}
$$

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Corollary 2. *If $1<p<2$, ξ is a nonnegative integrable function on Ω and $u,v\in W^{1,p}(\xi dx)$, then, by Hölder's inequality,*

$$
\begin{aligned}
\int_\Omega |\nabla u-\nabla v|^p\xi\,dx \;&=\; \int_\Omega |\nabla u-\nabla v|^p(|\nabla u|+|\nabla v|)^{\frac{(p-2)p}{2}}(|\nabla u|+|\nabla v|)^{\frac{(2-p)p}{2}}\xi\,dx \\
&\leq \left(\int_\Omega |\nabla u-\nabla v|^2(|\nabla u|+|\nabla v|)^{p-2}\xi\,dx\right)^{\frac{p}{2}}\left(\int_\Omega (|\nabla u|+|\nabla v|)^p\,\xi\,dx\right)^{\frac{2-p}{2}} \\
&\leq \left[\frac{1}{p-1}\int_\Omega (|\nabla u|^{p-2}\nabla u-|\nabla v|^{p-2}\nabla v)(\nabla u-\nabla v)\xi\,dx\right]^{\frac{p}{2}}\left[\int_\Omega (|\nabla u|+|\nabla v|)^p\xi\,dx\right]^{\frac{2-p}{2}}.
\end{aligned}
$$

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Theorem. *Suppose $1<p<\infty$ and, for each $i=1,2,\ldots$, $u_i\in W^{1,p}(\Omega,N)$ is a weak solution of*

$$\mathrm{div}\,(|\nabla u|^{p-2}\nabla u)\;+f_i=0$$

with $K\equiv\sup_i\|u_i\|_{W^{1,p}}+\sup_i\|f_i\|_{L^1}\;<\;\infty$. If $u_i\rightharpoonup u$ weakly in $W^{1,p}$, then $u_i\to u$ strongly in $W^{1,q}$ whenever $1<q<p$.

Proof : It suffices to prove that, for each $\delta\in(0,1]$,

$$\int_\Omega |\nabla u_i-\nabla u|^q\,dx \;=\; o(\delta)\;+\;o_\delta(\frac{1}{i})\;,$$

where we say an expression $M(\delta,i)$ is

$$o(\delta)\;\;\text{if}\;\;\lim_{\delta\to 0}\sup_i|M(\delta,i)|=0\;,\text{or}$$

$$o_\delta(\frac{1}{i}) \text{ if, for each } \delta, \lim_{i\to\infty} |M(\delta,i)| = 0 .$$

Denote by E^i_δ and F_δ the subsets

$$F_\delta = \{x \in \Omega : \; d(x) \le \delta\}; \quad E^i_\delta = \{x \in \Omega : |u_i(x) - u(x)| \ge \delta\},$$

where $d(x) = \mathrm{dist}(x, \partial\Omega)$ and $i = 1, 2, 3, \ldots$. Clearly $|(E^i_\delta)|$ is $o_\delta(\frac{1}{i})$ and $|F_\delta|$ is $o(\delta)$.

The lower-semicontinuity of $\int_\Omega |\nabla u|^p$ and the weak convergence $u_i \rightharpoonup u$ imply that

$$\|u\|_{W^{1,p}} \;\le\; K \;:=\; \sup_i \|u_i\|_{W^{1,p}} \; .$$

For $q < p$, the Hölder inequality gives

$$\int_{E^i_\delta \cup F_\delta} |\nabla u_i - \nabla u|^q \le 2^q K^q \left(|E^i_\delta|^{\frac{p-q}{p}} + |F_\delta|^{\frac{p-q}{p}} \right) \; = \theta_\delta(\frac{1}{i}) + o(\delta). \tag{1}$$

To show that $\int_{\Omega\setminus(E^i_\delta \cup F_\delta)} |\nabla u_i - \nabla u|^q \; = o(\delta) + o_\delta(\frac{1}{i})$, it suffices, by Hölder's inequality again, to show that

$$\int_{\Omega\setminus(E^i_\delta \cup F_\delta)} |\nabla u_i - \nabla u|^p \;=\; o(\delta) + o_\delta(\frac{1}{i}) \; .$$

For this, we define functions $\xi : \Omega \to [0, 1]$ and $\eta : \mathbf{R}^k \to \mathbf{R}^k$ by

$$\xi(x) = \min\{\frac{d(x)}{\delta}, 1\}, \quad x \in \Omega; \quad \eta(y) = \min\{\frac{\delta}{|y|}, 1\}y, \quad y \in \mathbf{R}^k.$$

It is direct to verify the following properties of ξ and η:

$$\xi|\partial\Omega = 0 \; , \quad \xi|F_\delta = 1 \; , \quad |\nabla\xi| \le \frac{1}{\delta} \; , \quad |\eta| \le \delta.$$

Using Corollaries 1 and 2, together with these properties of ξ and η, we obtain

$$\begin{aligned}
&\frac{1}{2}\int_{\Omega\setminus(E^i_\delta \cup F_\delta)} |\nabla u_i - \nabla u|^p \;\le\; \frac{1}{2}\int_{\Omega\setminus E^i_\delta} \xi|\nabla u_i - \nabla u|^p \\
&\le \int_{\Omega\setminus E^i_\delta} \xi \left(|\nabla u_i|^{p-2}\nabla u_i - |\nabla u|^{p-2}\nabla u \right) \cdot (\nabla u_i - \nabla u) \\
&= \int_{\Omega\setminus E^i_\delta} \xi|\nabla u_i|^{p-2}\nabla u_i \cdot \nabla(\eta \circ (u_i - u)) \;-\; \int_{\Omega\setminus E^i_\delta} \xi|\nabla u|^{p-2}\nabla u \cdot (\nabla u_i - \nabla u) \\
&\equiv \int_{\Omega\setminus E^i_\delta} I - \int_{\Omega\setminus E^i_\delta} II \; .
\end{aligned} \tag{2}$$

Now we look at each term. By the p-harmonic map equation and Hölder's inequality,

$$
\begin{aligned}
|\int_\Omega I\,| &= |\int_\Omega \xi|\nabla u_i|^{p-2}\nabla u_i\cdot\nabla\left(\eta\circ(u_i-u)\right)| \qquad (3)\\
&= |\int_\Omega \xi f_i\left(\eta\circ(u_i-u)\right) + \int_\Omega |\nabla u_i|^{p-1}\left(\eta\circ(u_i-u)\right)\cdot\nabla\xi|\\
&\le \delta K + K^{p-1}|F_\delta|^{1/p} = o(\delta).
\end{aligned}
$$

To estimate $\int_{E^i_\delta} I$, we note that on E^i_δ,

$$
\begin{aligned}
I &= \delta\xi|\nabla u_i|^{p-2}\nabla u_i\cdot\nabla\frac{u_i-u}{|u_i-u|}\\
&= \xi\delta\frac{|\nabla u_i|^{p-2}\nabla u_i^\alpha}{|u_i-u|}\left((\nabla u_i^\alpha-\nabla u^\alpha)-\frac{(u_i^\alpha-u^\alpha)(u_i^\beta-u^\beta)}{|u_i-u|^2}(\nabla u_i^\beta-\nabla u^\beta)\right)\\
&= \xi\delta\frac{|\nabla u_i|^{p-2}\nabla u_i^\alpha}{|u_i-u|}\left(\nabla u_i^\alpha-\frac{(u_i^\alpha-u^\alpha)(u_i^\beta-u^\beta)}{|u_i-u|^2}\nabla u_i^\beta\right)\\
&\quad+\xi\delta\frac{|\nabla u_i|^{p-2}\nabla u_i^\alpha}{|u_i-u|}\left(\nabla u^\alpha-\frac{(u_i^\alpha-u^\alpha)(u_i^\beta-u^\beta)}{|u_i-u|^2}\nabla u^\beta\right) = I'+I'',
\end{aligned}
$$

where the repeated indices α and β are summed from 1 to k. Note that $I'\ge 0$. As for I'', we have, by the Hölder inequality,

$$
|\int_{E^i_\delta} I''| \le 2\int_{E^i_\delta}|\nabla u_i|^{p-1}|\nabla u| \le 2K^{p-1}\left(\int_{E^i_\delta}|\nabla u|^p\right)^{1/p} = o_\delta(\frac{1}{i}). \qquad (4)
$$

For II, we use that $u_i \rightharpoonup u$ in $W^{1,p}$ and Hölder's inequality to get

$$
\begin{aligned}
|\int_{\Omega\setminus E^i_\delta} II| &\le |\int_\Omega \xi|\nabla u|^{p-2}\nabla u\cdot\nabla(u_i-u)| + |\int_{E^i_\delta}\xi|\nabla u|^{p-2}\nabla u\cdot\nabla(u_i-u)|\\
&\le o_1(\frac{1}{i}) + 2K\left(\int_{E^i_\delta}|\nabla u|^p\right)^{\frac{p-1}{p}} = o_\delta(\frac{1}{i}).
\end{aligned}
$$

Using that $I'\ge 0$ and combining (1)-(4), we obtain

$$
\begin{aligned}
\int_{\Omega\setminus(E^i_\delta\cup F_\delta)}|\nabla u_i-\nabla u|^p &\le \int_\Omega I - \int_{E^i_\delta} I'' - \int_{\Omega\setminus E^i_\delta} II\\
&\le |\int_\Omega I| + |\int_{E^i_\delta} I''| + |\int_{\Omega\setminus E^i_\delta} II| = o(\delta)+o_\delta(\frac{1}{i}).
\end{aligned}
$$

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Remarks. *In case N is the standard sphere $\mathbf{S}^k$, the weak limit u above is also a weak solution of the p-harmonic map equation.*

In fact, a map $u \in W^{1,p}(\Omega, \mathbf{S}^k)$ is, by an argument similar to Lemma 2.2 of [C], a weak solution of the p-harmonic map equation if and only if

$$\int_\Omega \left(|\nabla u|^{p-2}\nabla u \wedge u\right) \cdot \nabla\zeta \; dx \;=\; 0$$

for all $\zeta \in \mathcal{C}_0^\infty(\Omega, \mathbf{R}^k)$. Since $|u| \equiv 1$, this condition is clearly preserved under strong convergence in $W^{1,p-1}(\Omega, \mathbf{S}^k)$.

It remains an open problem whether a $W^{1,p}$ weak limit of p-harmonic maps is again p-harmonic, even for $p = 2$. For energy minimizers this is easy to verify because the convergence is then necessarily strong in $W^{1,p}$. Here the limit is also energy minimizing by [L]. Also, by the recent preprint [TW], a weak limit of p-harmonic maps will be p-harmonic in case the target N is a homogeneous space or in case the sequence consists of *stationary* maps (see [B]). But in the latter case it is unknown if the limit is also stationary.

For a *blowing-up sequence*, there is a convergence result similar to Theorem 1 above, and one may verify that the limit function satisfies the *blow-up equation.* (A different proof of this has been given by M. Fuchs [F]).

Suppose that, for $i = 1, 2, \ldots,$ $u_i \in W^{1,p}(\Omega, N)$ is a weak solution of the p-harmonic map equation,

$$\varepsilon_i \equiv \|\nabla u_i\|_{L^p} \;\to 0 \;\text{ as }\; i \to \infty \;,$$

and

$$v_i \;=\; \frac{u_i - \bar{u_i}}{\varepsilon_i} \text{ where } \bar{u_i} \;=\; (\text{meas }\Omega)^{-1} \int_B u_i \, dx \;.$$

If $v_i \rightharpoonup v$ weakly in $W^{1,p}$, then $v_i \to v$ strongly in $W^{1,q}$ whenever $1 < q < p$, and v is a weak $W^{1,p}(\Omega, \mathbf{R}^k)$ solution of the p-harmonic equation, i.e.

$$\int_\Omega |\nabla u|^{p-2}\nabla u \cdot \nabla\zeta \; dx \;=\; 0 \text{ for all } \;\zeta \in \mathcal{C}_0^\infty(\Omega, \mathbf{R}^k) \;.$$

Proof : First observe that

$$\begin{aligned}
&\Big|\int_\Omega |\nabla v_i|^{p-2}\nabla v_i\cdot\nabla\zeta\,dx\Big| \\
&= \varepsilon_i^{1-p}\Big|\int_\Omega |\nabla u_i|^{p-2}\nabla u_i\cdot\nabla\zeta\,dx\Big| \\
&= \varepsilon_i^{1-p}\Big|\int_\Omega |\nabla u_i|^{p-2}Q_{u_i}(\nabla u_i)\cdot\zeta\;dx\Big| \\
&\le \varepsilon_i c_N\|\zeta\|_{L^\infty}\int_\Omega |\nabla v_i|^p\,dx \;\to\; 0 \text{ as } i\to\infty\,.
\end{aligned}$$

Then we argue as in the proof of Theorem 1, now using $v_i - v$ in the definition of E^i_δ and in choosing the test function $\zeta = \xi\eta\circ(v_i - v)$. ■

John Hutchinson has kindly pointed out to us that the latter fact is used, without proof, in the argument on [HL, p.564].

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