Existence of Biharmonic Curves and Symmetric Biharmonic Maps

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The existence of curves and symmetric maps with minimal total tension is proved. Such curves and maps satisfy a class of fourth order differential equations.

1 Definitions of biharmonic maps and curves

Let N be a Riemannian manifold embedded into the Euclidean space R^m, $m \geq 2$, and Ω a smooth bounded domain in R^n, $n \geq 1$. Given maps $\varphi : \partial\Omega \to N$ and $\psi : \partial\Omega \to T_\varphi N$ (i.e., $\psi(x)$ is tangent to N at $\varphi(x)$ for $x \in \partial\Omega$), we look for an "optimal" map $u : \Omega \to N$ such that

$$u = \varphi,\ \frac{\partial u}{\partial n} = \psi \text{ on } \partial\Omega, \tag{1}$$

where n is the exterior normal direction of $\partial\Omega$. In other words, we look for a "best" way to extend the boundary value φ with the prescribed normal derivative ψ. Typical examples of Ω and N are the unit ball and the unit sphere, respectively. In this case, $\psi : \partial\Omega \to T_\varphi N$ means $\varphi(x) \cdot \psi(x) = 0$ for all $|x| = 1$.

With the given Dirichlet data φ, the most natural extension is perhaps the harmonic map. Recall that a map $u : \Omega \to N$ is harmonic if and only if its tension field $T(u)$ vanishes. In terms of the second fundamental form A of $N \subset R^m$, $T(u)$ can be expressed as

$$T(u) \equiv \Delta u - A(u)(\nabla u, \nabla u), \tag{2}$$

where u is considered as a vector valued function from Ω to R^m, Δu is the ordinary Laplacian of u, ∇u is the gradient of u, and $A(u)(\nabla u, \nabla u)$ is understood as the trace of A.

However, with the normal derivative being prescribed, it is easy to see that a harmonic extension does not generally exist. In fact, it was shown in [6] that for *almost all* $\varphi : \partial\Omega \to N$, there is a unique energy minimizing harmonic extension $u : \Omega \to N$; therefore, $\frac{\partial u}{\partial n}$ has been determined by φ. In this paper, we seek an extension u of φ with $\frac{\partial u}{\partial n} = \psi$ that is as close to a harmonic map as possible. Specifically, we consider the *total tension* of u

$$\mathcal{T}(u) = \int_\Omega |T(u)|^2 dx, \tag{3}$$

and try to find u as a minimum of $\mathcal{T}$.

Since $A(u)$ is the projection of Δu in the normal space of N at u, we have

$$|T(u)|^2 = |\Delta u|^2 - |A(u)(\nabla u, \nabla u)|^2. \tag{4}$$

Thus the natural class for the extensions is

$$\mathcal{C} = \left\{ u : u \in W^{2,2}(\Omega, N) \ \text{and satisfies } (1) \right\}, \tag{5}$$

where $W^{2,2}(\Omega, N)$ is the set of all $u : \Omega \to N \subset R^m$ with finite norm $||u||_{2,2}$, defined by

$$||u||_{2,2}^2 = \int_\Omega \left(|u|^2 + |\nabla u|^2 + |\nabla^2 u|^2 \right) dx, \tag{6}$$

Following the definition of Eells and Lemaire in [2], we will call a critical point of $\mathcal{T}(u)$ a *biharmonic map*, and when $n = 1$, a *biharmonic curve*. In 1986, Jiang [3] derived the first and second variation formulae of $\mathcal{T}$ and gave some examples of biharmonic maps, which include harmonic maps that are in $W^{2,2}$. However, there is no general existence result for biharmonic maps due to the fact that $\mathcal{T}$ is non-coercive. In Section 2 of this paper, we prove the existence of biharmonic curves. In Section 3, an existence result on axially symmetric biharmonic maps is proved.

Remark 1. A similar energy functional is $_\Omega|\nabla^2 u|^2 dx$ (or $\int_\Omega |\Delta u|^2 dx$), which is perhaps more interesting from an analytic point view. Chang, Wang and Yang [1] proved the partial regularity of critical points of $\int_\Omega |\Delta u|^2 dx$. Hardt, Mou and Wang [4] consider the partial regularity of minimizers of $\int_\Omega |\nabla^2 u|^2 dx$ under conditions different from that of [1]. Since these functionals are coercive, the existence of critical points and minimizers follows easily from direct method.

Remark 2. One might be interested in the path in $\mathcal{C}$ with least total curvature $\int_{u[-1,1]} |\kappa_g| ds$. However, such a path might not exist, or when it exists there might be infinitely many. Consider the case of plane curves. It is well known that if $\mathcal{C}$ contains a convex path, then $\int_{u[-1,1]} |\kappa_g| ds$ is constant for all convex curves (as they have same boundary conditions), and so each convex path is a minimum. While if $\mathcal{C}$ contains no convex path, then it could happen that no path would realize the infimum of the total curvature.

Remark 3. One might also be interested in the path of least total squared curvature $\int_{u[-1,1]} |\kappa_g|^2 ds$ of the Willmore type. This quantity might decrease to 0 as the length of $u[-1,1] \to \infty$. Therefore, unless we consider only paths with bounded length, the infimum 0 might never be realized. See [5] and the references therein.

2 Existence of biharmonic curves

For biharmonic curves $u : [-1, 1] \to N$, the condition (1) and definitions (5) (6) become

$$\{u(-1), u(1), u'(-1), u'(1)\} = \{p_1, p_2, v_1, v_2\} \in N \times N \times T_{p_1}N \times T_{p_2}N \quad (7)$$

$$\mathcal{C} = \left\{u : u \in W^{2,2}([-1,1], N) \text{ and satisfies } (7)\right\}, \quad (8)$$

$$||u||_{2,2}^2 = \int_{-1}^{1} \left(|u|^2 + |u'|^2 + |u''|^2\right) dt. \quad (9)$$

The total tension of $u \in \mathcal{C}$ is

$$\mathcal{T}(u) = \int_{-1}^{1} |T(u)|^2 dt = \int_{-1}^{1} \left(|u''|^2 - |A(u)(u', u')|^2\right) dt. \quad (10)$$

We prove the following existence result.

Theorem 1. *Given boundary data as in (7) such that the admissible set $\mathcal{C} \neq \emptyset$, the total tension $\mathcal{T}(u)$ has a minimum in $\mathcal{C}$.*

The proof of this theorem is a standard direct method. The key ingredients are Lemmas 1, 2, which are proved later in this section. We conjecture that Lemmas 1, 2 continue to hold for $n \geq 2$, which would imply the existence of biharmonic maps.

For $u \in \mathcal{C}$ and $q > 0$, we denote $\mathcal{D}_q(u) = \int_{-1}^{1} |u'|^q dt$. The following lemmas will be proved later.

Lemma 1. *For every $u \in \mathcal{C}$,*

$$|u(t) - p_1| + |u(t) - p_2| \leq \int_{-1}^{1} |u'(s)| ds \leq \sqrt{2}\, \mathcal{D}_2(u)^{1/2}, \quad (11)$$

$$\mathcal{D}_2(u) \leq 2\left(|v_1|^2 + |v_2|^2\right) + 4\mathcal{T}(u), \quad (12)$$

$$\mathcal{D}_4(u) \leq 2\left(|v_1|^2 + |v_2|^2\right)^2 + 32\mathcal{T}(u)^2. \quad (13)$$

Lemma 2. *Suppose $u \in \mathcal{C}$ and $\mathcal{T}(u) \leq K$, then*

$$||u||_{2,2}^2 \leq M, \quad (14)$$

where M is a constant depending only on K, N and $\{p_1, p_2, v_1, v_2\}$.

Proof of Theorem 1. We use the direct method in calculus of variations. Suppose $\{u^k\}$ is a minimizing sequence such that $\mathcal{T}\left(u^k\right) \to \inf_{u\in\mathcal{C}} \mathcal{T}(u)$ as $k \to \infty$. By Lemma 2, the sequence $\{u^k\}$ is bounded in $W^{2,2}$. Therefore, a subsequence exists and weakly converges to some $u \in \mathcal{C}$. It is easy to see that $\mathcal{T}(u)$ is lower semicontinuous with respect to the weak convergence in $W^{2,2}$. So u is a minimum. $\square$

Proof of Lemma 1. For $t \in [-1, 1]$, by the fundamental theorem of calculus, we have

$$\begin{aligned} &|u(t) - p_1| + |u(t) - p_2| \\ &\leq \int_{-1}^{t} |u'(s)|ds + \int_{t}^{1} |u'(s)|ds = \int_{-1}^{1} |u'(s)|ds \\ &\leq \sqrt{2}\, \mathcal{D}_2(u)^{1/2}. \end{aligned} \tag{15}$$

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This proves (11). Now using the fact that u' is a tangent vector, which implies that $u' \perp A(u)$, we have

$$\begin{aligned} |u'(t)|^2 &= |v_1|^2 + \int_{-1}^{t} 2u'(s)\, u''(s)\, ds \\ &= |v_1|^2 + \int_{-1}^{t} 2u'(s)\, T(u(s))\, ds \\ &\leq |v_1|^2 + 2\int_{-1}^{t} |u'(s)||T(u(s))|ds. \end{aligned} \tag{17}$$

Similarly, we have

$$|u'(t)|^2 \leq |v_2|^2 + 2\int_{t}^{1} |u'(s)||T(u(s))|ds. \tag{18}$$

Averaging the above two estimates, we have

$$|u'(t)|^2 \leq \frac{1}{2}\left(|v_1|^2 + |v_2|^2\right) + \int_{-1}^{1} |u'(s)||T(u(s))|ds \tag{19}$$

Integrating (19) over $[-1,1]$ and applying Schwarz's inequality to the integral, we get

$$\mathcal{D}_2(u) \leq \left(|v_1|^2 + |v_2|^2\right) + \frac{1}{2}\mathcal{D}_2(u) + 2\mathcal{T}(u). \tag{20}$$

Solving for $\mathcal{D}_2(u)$ gives (12).

Next squaring the both sides of (19), using that $(a+b)^2 \leq 2a^2 + 2b^2$, and then using that $\mathcal{D}_2(u) \leq \sqrt{2}\,\mathcal{D}_4(u)^{1/2}$, we have

$$\begin{aligned} |u'(t)|^4 &\leq \frac{1}{2}\left(|v_1|^2 + |v_2|^2\right)^2 + 2\mathcal{D}_2(u)\,\mathcal{T}(u) \\ &\leq \frac{1}{2}\left(|v_1|^2 + |v_2|^2\right)^2 + 2\sqrt{2}\mathcal{D}_4(u)^{1/2}\,\mathcal{T}(u) \\ &\leq \frac{1}{2}\left(|v_1|^2 + |v_2|^2\right)^2 + \frac{1}{4}\mathcal{D}_4(u) + 8\mathcal{T}(u)^2 \end{aligned} \tag{21}$$

Integrating this estimate over $[-1,1]$, we get

$$\mathcal{D}_4(u) \leq \left(|v_1|^2 + |v_2|^2\right)^2 + \frac{1}{2}\mathcal{D}_4(u) + 16\mathcal{T}(u)^2. \tag{22}$$

Solving for $\mathcal{D}_4(u)$, we get (13).□

Proof of Lemma 2. Assuming $u \in \mathcal{C}$ and $\mathcal{T}(u) \leq K$. By Lemma 1, we see that $||u||_\infty$ is bounded in terms of K and $\{p_1, p_2, v_1, v_2\}$. It follows that $||A(u)||_\infty$ is bounded in terms of K and $\{p_1, p_2, v_1, v_2\}$ and N. Therefore by (10),

$$\mathcal{T}(u) \geq \int_{-1}^{1} |u''|^2 dt - ||A(u)||_\infty\, \mathcal{D}_4(u), \tag{23}$$

which implies that

$$\int_{-1}^{1} |u''|^2 dt \leq \mathcal{T}(u) + ||A(u)||_\infty\, \mathcal{D}_4(u). \tag{24}$$

By Lemma 1 again, $\mathcal{D}_4(u)$, and therefore $\int_{-1}^{1} |u''|^2 dt$, is bounded in terms of K, N and $\{p_1, p_2, v_1, v_2\}$.□

3 Existence of symmetric biharmonic maps

In this section we consider the special case when $\Omega = B$, the unit ball in R^n and $N = S^n \in R^{n+1}$, $n \geq 2$. In this case the total Hessian of $u \in W^{2,2}(B, S^n)$ is

$$\mathcal{T}(u) = \int_B \left(|\Delta u|^2 - |\nabla u|^4\right) dx, \tag{25}$$

and the Euler-Lagrange equation satisfied by a biharmonic map $u : B \to S^n$ is

$$\Delta^2 u + 2\nabla\left(|\nabla u|^2 \nabla u\right) + (3\Delta(\nabla u)\cdot\nabla u + |\Delta u|^2)u = 0. \tag{26}$$

To derive this, we first get the Euler equation $\Delta^2 u + 2\nabla\left(|\nabla u|^2\nabla u\right) = \lambda\, u$ with a Lagrange multiplier λ. Using the fact $1 = u\cdot u$, we find and simplify λ as

$$\lambda = \left(\Delta^2 u + 2\nabla\left(|\nabla u|^2\nabla u\right)\right)\cdot u = -\left(3\Delta(\nabla u)\cdot\nabla u + |\Delta u|^2\right). \tag{27}$$

A map $u : B \to S^n$ is said to be *axially symmetric* if there is a map f: $S^{n-1} \to S^{n-1} \subset S^n$ and a function $\varphi : [0,1] \to [0,\pi]$ such that for $x \in B\setminus\{0\}$,

$$u\,(x) = \left(f\,(\theta)\,\sin\varphi\,(r)\,,\,\cos\,\varphi\,(r)\right), \tag{28}$$

where $r = |x|$ and $\theta = x/|x|$.

We assume that $f\,(\theta) : S^{n-1} \to S^{n-1}$ is a harmonic map, that is,

$$\Delta_\theta f + |\nabla_\theta f|^2 f = 0. \tag{29}$$

As for φ, we assume that it satisfies the boundary conditions

$$\varphi\,(0) = \varphi'\,(0) = 0,\ \varphi\,(1) = a,\ \varphi'\,(1) = b, \tag{30}$$

and that the radial function $\varphi\,(|x|) : B \to R$ belongs to the space $W^{2,2}\,(B)$.

We calculate

$$\begin{aligned}
u_r &= u_\varphi \varphi' = (f\cos\varphi,\ -\sin\varphi)\,\varphi' \\
u_{rr} &= u_\varphi\varphi'' - u\varphi'^2 \\
\Delta_\theta u &= -|\nabla_\theta f|^2\,(f\,\sin\varphi,\,0) \\
|\nabla u|^2 &= \varphi'^2 + \frac{|\nabla_\theta f|^2}{r^2}\sin^2\varphi
\end{aligned}$$

$$\begin{aligned}
\Delta\varphi &= \varphi'' + \frac{n-1}{r}\varphi' \\
\Delta u &= u_\varphi\Delta\varphi - u\varphi'^2 - \frac{|\nabla_\theta f|^2}{r^2}\,(f\sin\varphi, 0) \\
T\,(u) &= \Delta u + |\nabla u|^2 u = u_\varphi\left[\Delta\varphi - \frac{|\nabla_\theta f|^2}{2r^2}\sin 2\varphi\right]
\end{aligned}$$

$$\mathcal{T}\,(u) = \int_B |\Delta\varphi - \frac{|\nabla_\theta f|^2}{2r^2}\sin 2\varphi|^2 dx \tag{31}$$

Lemma 3. *If φ is a minimum of (31) satisfying (30), then the axially symmetric map u defined by (28) is a biharmonic map with boundary data*

$$u|_{\partial B} = (f\,(\theta)\sin a,\, \cos a)\,,\ \frac{\partial u}{\partial n}|_{\partial B} = (f\ \,(\theta)\cos\, a,\ -\sin\, a)\, b. \qquad (32)$$

Proof. Suppose φ is a minimum of (31). Consider a variation φ_t of φ with $\frac{d}{dt}\varphi_t|_{t=0} = \eta \in C\,[0,1]$ with $\eta\,(0) = \eta\,(1) = 0$. Let u_t be defined as in (28) with φ_t. We have

$$\begin{aligned} 0 \ &= \frac{d}{dt}\mathcal{T}\,(u_t)\,|_{t=0} \qquad (33)\\ &= 2\int_B \left(\Delta\varphi - \frac{|\nabla_\theta f|^2}{2r^2}\sin 2\varphi\right)\left(\Delta\eta - \frac{|\nabla_\theta f|^2}{r^2}\cos 2\varphi\ \eta\right)dx \\ &= 2\int_B \left[\Delta T\,(\varphi) - \frac{\alpha}{r^2}\cos 2\varphi\Delta\varphi + \frac{\beta}{4r^4}\sin\, 4\varphi\right]\eta dx, \end{aligned}$$

where $T\,(\varphi) = \Delta\varphi - \frac{\alpha}{2r^2}\sin 2\varphi$, Δ is the Laplacian on $\varphi\,(|x|)$, and

$$\alpha = \frac{1}{|S^{n-1}|}\int_{S^{n-1}} |\nabla_\theta f|^2 d\sigma,\ \beta = \frac{1}{|S^{n-1}|}\int_{S^{n-1}} |\nabla_\theta f|^4 d\sigma \qquad (34)$$

are the average values of $|\nabla_\theta f|^2$ and $|\nabla_\theta f|^4$, respectively. Thus (33) implies that

$$\Delta T\,(\varphi) - \frac{\alpha}{r^2}\cos 2\varphi\Delta\varphi + \frac{\beta}{4r^4}\sin\, 4\varphi = 0. \qquad (35)$$

It is not hard to see that this is the same as (26) for axially symmetric map. So a critical point φ of (31) defines biharmonic map.□

Theorem 2. *Suppose $n \geq 5$, $f\colon S^{n-1} \to S^{n-1}$ is harmonic, and a, b are two numbers, then there is an axially symmetric biharmonic map $u\colon B \to S^n$ as in (28).*

Proof. The method of proof is again the direct method in calculus of variations. By Lemma 3, it is equivalent to showing that (31) has a critical point. The key ingredient here is to show that if $\{\varphi^k\}$ is a minimizing sequence of (31), then $\int_B |\Delta\varphi^k|^2 dx$ will be bounded. Indeed, from (31) we get

$$\mathcal{T}\,(u)\ = \int_B |\Delta\varphi - \frac{|\nabla_\theta f|^2}{2r^2}\sin 2\varphi|^2 dx \qquad (36)$$

$$\geq \int_B \left[\frac{1}{2}|\Delta\varphi|^2 - \left(\frac{|\nabla_\theta f|^2}{2r^2} \sin 2\varphi \right)^2 \right] dx$$

$$\geq \int_B \frac{1}{2}|\Delta\varphi|^2 dx - \beta \int_B \sin^2\varphi \cos^2\varphi \, r^{n-5} dx,$$

where β is defined in (34). Since $n \geq 5$,

$$\int_B |\Delta\varphi|^2 dx \leq 2\mathcal{T}(u) + 2\beta|S^{n-1}| \int_0^1 \sin^2\varphi \cos^2\varphi \, r^{n-5} dr \leq 2\mathcal{T}(u) + \frac{2\beta|S^{n-1}|}{n-4}. \tag{37}$$

This implies that $\left\{\varphi^k(|x|)\right\}$ is bounded in $W^{2,2}(B, R^m)$ and so it has a subsequence weakly converging to some φ, which must be a minimum by lower semicontinuity of (31).□

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