1. Introduction. Proper knot theory deals with proper embeddings of the real line into open 3-manifolds. Two such embeddings *f* and *g* are said to be equivalent if there is a proper isotopy connecting the two embeddings. This is, in general, a non-ambient classification theory. For example, there is one equivalence class of smooth (or p. l.) proper knots in R^3 (e. g., see page 183, exercise 9 of [5] or reference [2]).

Churchard and Spring have obtained classification theorems for smooth proper knots in open solid handlebodies and Klein bottles (of countable genus) and for $\mathsf{F}^2 \times \mathsf{R}$ (where F is a smooth, closed surface) ([2] and [3]). It was shown that, up to equivalence and orientation, there is a unique smooth proper knot in open solid handlebodies and Klein bottles. It was also shown that smooth proper knot equivalence classes in $\mathsf{S}^2 \times \mathsf{R}$ are completely determined by the ends to which the proper knot in those equivalence classes run. In [7], the author showed that two p. l. proper knots that are equivalent under a p. l. proper isotopy are connected by a locally flat p. l. proper isotopy. Hence, the smooth and p. l. classification of proper knots are very similar.

In [6], the author modified Churchard and Spring's techniques to show that topological proper knots in R^3 which are tame at a point are all equivalent. In fact, those theorems apply to proper knots which pierce a disk at one of its points. However, it is still unknown whether there are any inequivalent topological proper knots in R^3.

The main results of this paper are the following:

1) Theorem 3.1, which states that if *f* is a proper knot running between the opposite ends of $\mathsf{D}^2 \times \mathsf{R}$, then, up to orientation, *f* is equivalent to the proper knot which runs along $0 \times \mathsf{R}$. The idea, which was suggested to the author my Bob Daverman and Ric Ancel, is to use an equivalence by a "plunger" technique, as suggested by Figures 1, 2 and 3.

2) Theorem 3.4, which states that if the image of *f* pierces a disk at all of its points and is locally homogenous then *f* is equivalent to a p. l. proper knot and

3) Theorem 3.11, which states that if *f* is a proper knot whose set of wild points has no limit point and runs between a sphere end and a collared end and pierces a disk at each of its points then *f* is equivalent to a p. l. (or smooth) proper knot.

It is an easy consequence of Theorem 3.1 that a proper knot which can be "engulfed" to run between the opposite ends of an embedded $\mathsf{D}^2 \times \mathsf{R}$ is equivalent to a p. l. proper knot. Hence any "non-trivial" proper knot in R^3 would have to fail to pierce a disk at each of its points and fail to run between the opposite ends of a properly embedded $\mathsf{D}^2 \times \mathsf{R}$.

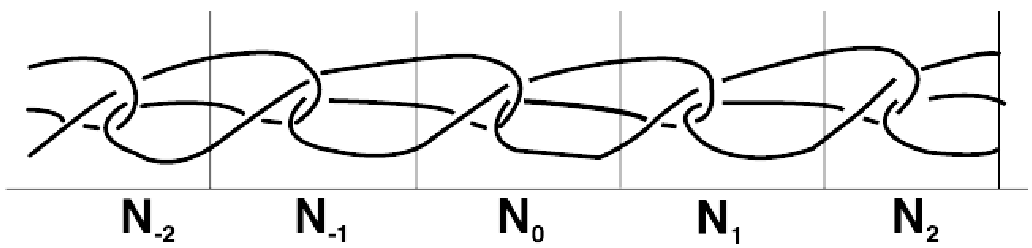
2. Preliminaries. The notations from reference [3] will be followed. Unless otherwise stated, the target 3-manifolds for the proper knots will be piecewise linear and non-compact. A map $f : X \to Y$ is called *proper* if for all compact $C \subset Y$, $f^{-1}(C)$ is compact in X. A proper knot in a 3-manifold M^3 is a topological proper embedding $f : \mathsf{R}^1 \to M^3$. Two proper knots *f* and *g* will be said to be *equivalent* if there exists a topological proper map $F : \mathsf{R}^1 \times [0,1] \to M^3$ so that $f = F_0$, $g = F_1$ and that F_t is an embedding for each $t \in [0,1]$. F is called a *proper isotopy*. If F is a piecewise linear (p. l.) map, we then say that *f* and *g* are p. l.-equivalent.

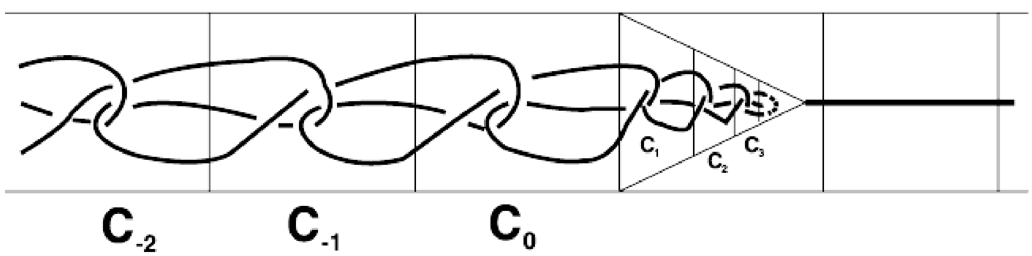
In this paragraph, we review the definition of an *end* of a non-compact manifold M: let $\{K_i\}$ be a compact exhaustion of M (that is, $M = \cup_i K_i$, $i \in \{1,2,3\ldots\}$, each K_i is compact, and $K_i \subset int(K_{i+1})$. Now form a sequence $U_1 \supset U_2 \supset U_3 \supset \ldots$ where each U_i is a path component of $M - K_i$ and each U_i has non-compact closure. Note that each U_i is open, has compact frontier and $\cap_i U_i = \emptyset$. If another such sequence of open sets V_i are generated from another compact exhaustion of M, we say that $\{U_i\}$ and $\{V_i\}$ are *equivalent* if they are cofinal; that is, for all i there exists a j so that $V_j \subset U_i$ and that for all m, there exists an n so that $U_n \subset V_m$. An equivalence class of such sequences is called an end of M, and the set of ends of M will be denoted by $e(M)$. An end $\Gamma \in e(M)$ will be called a *collared end* if there exists

$\{V_i\} \in \Gamma$ and an index j so that V_j is p. l. homeomorphic to $W \times [0,\infty)$ where W is a p. l., closed connected surface. A W-end denotes a collared end with collar surface W. The set corresponding to a collar $W \times [0,\infty)$ associated with a collared end Γ will be denoted by E.

In this paragraph, we state what is meant by a proper knot *running between* ends Γ_1 and Γ_2. Let $g : M \to N$ be a proper map between manifolds and let $\{U_i\} \in \Gamma_M \in e(M)$, $\{V_j\} \in \Gamma_N \in e(N)$. *$g$ sends end Γ_M to Γ_N* for all j, there exists an i so that $g(U_i) \in V_j$. g induces a well defined map $\widehat{g} : e(M) \to e(N)$ where for all $\Gamma \in e(M)$, $\widehat{g}$ sends Γ to $g(\Gamma)$. Note that $e(\mathsf{R}^1)$ is denoted by $\{-\infty,\infty\}$. Given a proper knot $f : \mathsf{R}^1 \to M$ and ends Γ_1 and Γ_2 (possibly the same end), we say that f runs between Γ_1 and Γ_2 if $\widehat{g}(\{-\infty,\infty\}) = \{\Gamma_1,\Gamma_2\}$. It is clear that if f and g are equivalent proper knots, then f and g run between the same ends.

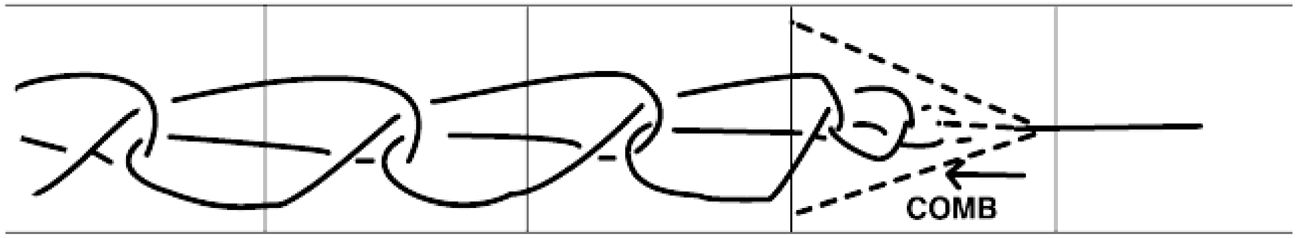
3. Classification and Smoothing Theorems. Let N be the non-compact manifold $\mathsf{D}^2 \times \mathsf{R}^1$, where D^2 is a 2-disk. We think of N as being the union of "cans" $N_i = \mathsf{D}^2 \times [i, i+1]$, $i \in \{..-2, -1, 0, 1, 2...\}$ which are glued together in a standard way. We can use cylindrical coordinates (x,r,θ), where $x \in \mathsf{R}^1$, $r \in [0, 1]$, $\theta \in [0, 2\pi)$ to describe the location of points in N. Denote the two ends of N by $-\infty$ and ∞.

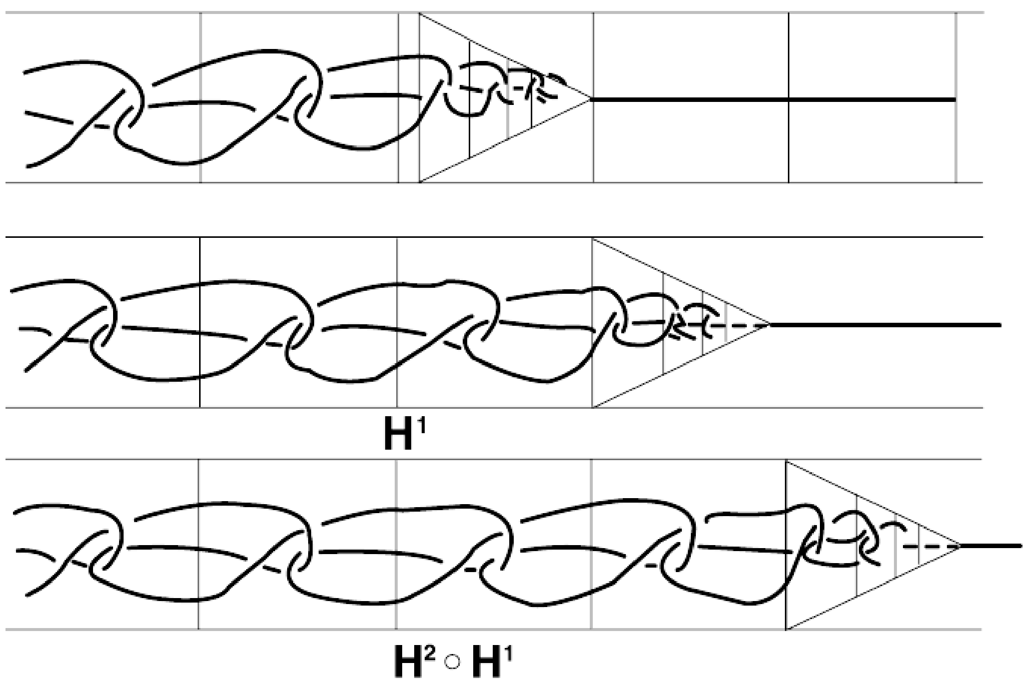




Theorem 3.1. *Let f be a proper knot running from $-\infty$ to ∞ in N. Then f is equivalent to the proper knot $h : \mathsf{R}^1 \to N$ where $h(x) = (x,0,0)$ for all $x \in \mathsf{R}^1$. That is, up to orientation and equivalence, there is a unique proper knot running between the opposite ends of $\mathsf{D}^2 \times R^1$.*

Proof. The idea of the proof is conveyed in Figures 1, 2 and 3. One can think of a proper knot running between the opposite ends of N as "piercing a disk at infinity."





Let f be a proper knot running between $-\infty$ and ∞ in N. With no loss of generality, we can assume that for all $k \in \{-2, -1, 0, 1, 2..\}$, $k = sup\{x \in \mathsf{R}^1 | f(x) \in N_{k-1}\}$. Consider the solid "cone" $C \subset N$ with missing "tip", where the base of C is $\mathsf{D}^2 \times \{0\}$, and whose curved surface is given by the image of $\partial \mathsf{D}^2 \times \mathsf{R}^1$ (here, ∂ denotes "boundary of") under the map $\gamma : N \to N$ by

$$\gamma(x,r,\theta) = \begin{cases} (x,r,\theta) \text{ for } x < 0 \\ (\frac{x}{x+1}, \frac{r}{x+1}, \theta) \text{ for } x \geq 0 \end{cases}.$$

The interior of C is homeomorphic to $(\mathsf{D}^2 - \partial\mathsf{D}) \times [0,\infty)$, and we let C_i denote the image of N_i under γ. Let $\phi: (-\infty, 1) \to \mathsf{R}^1$ be defined by $\phi(x) = \begin{cases} x, x \leq 0 \\ \frac{x}{1-x}, 0 < x < 1 \end{cases}$. We now consider the proper knot $g : \mathsf{R}^1 \to N$ defined by $g(x) = \begin{cases} \gamma \circ f \circ \phi(x) \text{ for } x < 1 \\ (x,0,0) \text{ for } x \geq 1 \end{cases}$. Clearly, g is a proper knot. Claim One: g is equivalent to the "trivial "proper knot h. Proof of Claim One: One can use a meridional disk, say $\mathsf{D}^2 \times \{2\}$, to comb (or push, as a plunger in a syringe) the non-trivial parts of g to $-\infty$. This is Proposition 2.2 of reference [3], equations with coordinates can be found in Lemma 2.2 of reference [7] See Figure 2.

Claim Two: g is equivalent to the proper knot f. Proof of Claim Two: Please refer to Figure 3. There is an ambient isotopy $H^1 : N \times [0,1] \to N$ that takes C_1 to N_1 so that $H^1_1(C_1) = \gamma^{-1}(C_1)$. Furthermore, we have H^1 taking $\cup_{i\geq 2} C_i$ into N_2 as well as fixing N_j, for all $j \leq 0$. Note that $H^1_1 \circ g$ defines a proper knot where $Image(H^1_1 \circ g) \cap N_1 = \gamma^{-1}(\gamma \circ f \circ \phi) \cap N_1 = Image(f \circ \phi) \cap N_1 = \text{Image}(f) \cap N_1$. We now compose $H^1_1 \circ g$ with an isotopy ϕ_1 which reparametrizes R^1 as follows: ϕ_1 takes $(-\infty, 0]$ to $(-\infty, 0]$ by the identity map, $(0, \phi^{-1}(1)]$ to $(0,1]$ via the map $\phi(x) = \frac{x}{1-x}$, $(\phi^{-1}(1), 1)$ to $(1, 2)$ and $[1,\infty)$ to $[2,\infty)$. Let $g_1 = H^1_1 \circ g \circ \phi_1$.

Next we choose an ambient isotopy H^2 that fixes N_j for all $j \leq 1$, takes $H^1_1(C_2)$ to N_2 by $H^2_1(H^1_1(C_2)) = \gamma^{-1}((H^1_1)^{-1}(H^1_1(C_2)))$, and takes $H^1_1(\cup_{i\geq 3} C_i)$ into N_3. We now compose $H^2_1 \circ H^1_1 \circ g_1$ with a reparametrization isotopy ϕ_2 (where ϕ_2 takes $(-\infty, 1]$ to $(-\infty, 1]$ by the identity map, $(1, \phi_1^{-1} \circ \phi^{-1}(2)]$ to $(1,2]$ via the map $(\phi_1 \circ \phi)$, $(\phi_1^{-1} \circ \phi^{-1}(2), 2)$ to $(2,3)$ and $[2,\infty)$ to $[3,\infty)$) to obtain a proper knot g_2. Note that $f|(-\infty, 2] = g_2|(-\infty, 2]$. We repeat this process to obtain for all positive integers k, a proper knot $g_k = H^k_1 \circ g_{k-1} \circ \phi_k$, where $H^k_1(H^{k-1}_1(H^{k-2}_1(\ldots(H^1_1(C_k))..)) = \gamma^{-1}((H^k_1 \circ H^{k-1}_1 \ldots \circ H^1_1)^{-1}(H^k_1 \circ H^{k-1}_1 \ldots \circ H^1_1)(C_k)$ and ϕ_k is an appropriate reparametrization isotopy. Note that $f|(-\infty, k] = g_k|(-\infty, k]$.

Now concatenate the proper isotopies $H^1, H^2, H^3, \ldots H^k \ldots$ in the following way: let σ_k be a map $\sigma_k : [0,1] \times [\frac{k-1}{k}, \frac{k}{k+1}]$ $(k \in \{1,2,3,\ldots\})$ defined by $\sigma_k(t) = (\frac{1}{k(k+1)})t + \frac{k-1}{k}$ for $t \in [0,1]$. We can then define a proper isotopy $H : [0,1] \times \mathsf{R}^1 \to N$ by $H(\sigma_k(t), x) = H^k(t,x)$ for $t \in [\frac{k-1}{k}, \frac{k}{k+1})$ and $H(1,x) = f(x)$.

It is immediate that H is continuous on $[0,1) \times \mathsf{R}^1$. To see that H is continuous on $[1-\epsilon, 1] \times \mathsf{R}^1$ (ϵ small and positive), let $x \in \mathsf{R}^1$ be given. There exists a positive integer k so that $x \in [k, k+1]$. Note that for all integers $n \geq 1$, and $y \geq x$, $g_{k+1}(y) = g_{k+n}(y) = f(y)$. Hence H is continuous. We now check that H is proper: given $X \subset N$, X compact, there exists integers i and j so that $X \subset \cup_{i\leq k\leq j} N_k$. Note that for $k \geq j+1$, $(H^{k+1})^{-1}(X) = f^{-1}(X)$. So, $H^{-1}(X) =$

$(H|\{[\frac{k+1}{k+2}, 1] \times \mathsf{R}^1\})^{-1}(X) \cup (H|\{[0, \frac{k+1}{k+2}] \times \mathsf{R}^1\})^{-1}(X) =$

$f^{-1}(X) \cup (H|\{[0, \frac{k+1}{k+2}] \times \mathsf{R}^1\})^{-1}(X)$ which is compact since it is the union of two compact sets. □

We get the immediate corollary:

Corollary 3.2. *Let f be a proper knot in a 3-manifold M. If the image of f runs between the opposite ends of a $\mathsf{D}^2 \times \mathsf{R}^1$ which is properly embedded in M, then f is equivalent to a piecewise linear proper knot.*

Corollary 3.2 leads to a classification theorem for proper knots running between the opposite ends of the manifold $\mathsf{S}^2 \times \mathsf{R}^1$.

Corollary 3.3. *Up to orientation and equivalence, there is a unique proper knot running between the opposite ends of* $\mathsf{S}^2 \times \mathsf{R}^1$.

Proof. Let f be a proper knot running between the opposite ends of $\mathsf{S}^2 \times \mathsf{R}^1$. By using a general position argument with a p. l. approximation of f, we can find some "collar line" proper knot g whose image is $* \times \mathsf{R}^1$, $* \in \mathsf{S}^2$. Hence, the image of f is properly embedded in the properly embedded $\mathsf{D}^2 \times \mathsf{R}^1$ which is formed by the complement of a regular neighborhood of the image of g. Hence f is equivalent to a p. l. proper knot.□

Note that it is still an open question whether every proper knot that runs to and from the same end of $\mathsf{S}^2 \times \mathsf{R}^1$ is equivalent to a p. l. proper knot; however such a "non-smoothable" proper knot would have to be wild enough to fail to pierce a disk at any of its points and badly embedded enough to fail to run in between the opposite ends of a properly embedded $\mathsf{D}^2 \times \mathsf{R}^1$.

We now use the work of Bothe [1] to give some sufficient conditions for a proper knot to be smoothable. We say that a proper knot has a *normal neighborhood at* $f(x)$ if there exists a 3-ball B containing x in its interior such that the image of f intersects ∂B in a two point set and f pierces ∂B at both intersection points. We say that f is *locally homogeneous* in M if, given any points $f(x)$ and $f(y)$, there are subarcs of the image of f, L_x and L_y which contain $f(x)$ and $f(y)$ in their interiors and an orientation preserving homeomorphism $h : M \to M$ such that $h(L_x) = L_y$ and $h(x) = y$. f is *homogeneous* if given any pair of points $(f(x), f(y))$, there is an orientation preserving homeomorphism $h : M \to M$ such that h fixes the image of f setwise and $h(f(x)) = f(y)$. It is easy to see that if f is homogenous, f is locally homogenous.

Theorem 3.4. *Suppose f is a proper knot whose image is locally homogenous in M and pierces a disk at one of its points(and therefore all of its points). Then f is equivalent to a p. l. proper knot.*

Proof. We suppose that $K_1, K_2, \ldots. K_n, ..$ is a compact exhaustion for M. With no loss of generality, we can assume that $f(0) \in K_1$, and that for all positive integers i, $i = min\{x \in (0,\infty) | f(x) \in \partial K_i\}$ and $-i = max\{x \in (-\infty, 0) | f(x) \in \partial K_i\}$. Because the image of f is locally homogeneous and pierces a disk at each of its points, the work of Bothe [1] shows that each point of $f(x)$ has a normal neighborhood of arbitrarily small size. Choose a covering B_1 of $f[-1,1]$ which includes normal neighborhoods B_{-1}, B_1 of $f(-1)$ and $f(1)$ respectively. In 2.12 and 2.13 of [1] (in the proof of Theorem 1), it is shown that the elements of B_1 can be chosen to be disjoint if they have subarc of f in common, and to intersect such that their boundaries meet in a single simple closed curve. Call such a collection "tubular." Let $x_1 = min\{x \in \mathsf{R} | f(x) \in B_{-1}\}$ and $y_1 = max\{x \in \mathsf{R} | f(x) \in B_1\}$. Then $f([-1,1])$ misses $f((-\infty, x_1]) \cup f([y_1, \infty))$ by a set distance ϵ_1. We can then obtain a new minimal covering of $f([x_1, y_1])$ by normal neighborhoods which are of size less that $\epsilon_1/3$. Call the union of the covering normal neighborhoods N_1'.

Now consider the arcs $f([y_1, 2])$ and $f([-2, x_2])$. These arcs have a covering B_2 by normal neighborhoods of size less that $\epsilon_1/3$ which include the normal neighborhoods B_{-2} and B_2 of $f(-2)$ and $f(2)$. N_1' and the elements of B_2 can be modified so as to constitute a tubular collection of normal neighborhoods of $f([-2, x_1])$ and $f([y_1, 2])$. Define x_2 and y_2 as before, and note that $f([-2,2])$ misses $f((-\infty, x_2]) \cup f([x_2, \infty))$ by a set distance ϵ_2. So we can modify the tubular collection of coverings again so as to ensure that the size of all elements of B_2 are less than $min\{\epsilon_1/3, \epsilon_2/3\}$. We can modify N_1' again at its "end elements" (the elements that cover $f(x_1)$ and $f(y_1)$) to get N_1 such that the covering of $f([-2, 2])$ remains tubular.

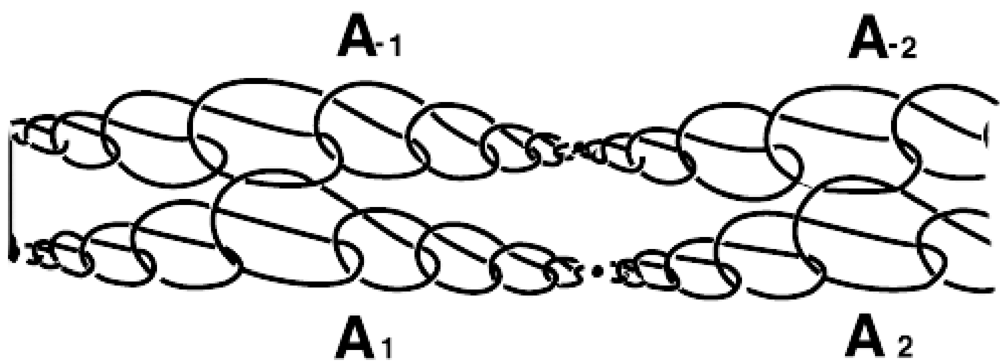
This process can be repeated for all $f([-i, i] \subset K_i$ $(i \geq 3)$. Note that the building of the tubular cover for $f([-i, x_{i-1}])$ and $f([y_{i-1}, i])$ does not require modifying N_j for $j \leq i-2$. Hence, in a manner similar to the way a "defining torus" was fit around a simple closed curve in [1], we can fit a properly embedded $D^2 \times R^1$ where each $D^2 \times [-i - \delta_i, i + \delta_i]$ can be identified with N_i $(\delta_i > 0)$. Then by Theorem 3.1, f is equivalent to the centerline of this embedded $D^2 \times R^1$ and thus is equivalent to a p. l. proper knot.

We now present an analogy to Proposition 3.3 of [3] for topological proper knots.

Corollary 3.5. *Let f be a proper knot in a 3-manifold M which runs to a sphere end* Γ. *Then f is equivalent to a proper knot g which follows a collar ray* $* \times [a, \infty)$ $(a > 0)$ *of the collar E associated with* Γ. *Furthermore, it can be assumed that the proper isotopy connecting f to g fixes f on* $M - E$.

Proof. By adjustment with an ambient isotopy, it can be assumed that the image of f misses some collar line $w \times [0, \infty)$ of E. Then one can take the complement of the regular neighborhood of $w \times [0, \infty)$ within Γ to obtain a properly embedded $D^2 \times [0, \infty) \subset E \subset M$ that contains the image of a "selected half" of f, say $f|[0, \infty)$. The proper isotopy discussed in Claim Two of Theorem 3.1 can be used to properly isotope f to a proper knot g which follows the centerline of the properly embedded $D^2 \times [a, \infty)$ for some $a > 0$. Note that this isotopy is topological and could well take a p. l. proper knot to a wild one. □

Example 3.6. *f need not run between the ends of a properly embedded* $D^2 \times R^1 \subset M$ *in order to be smoothable.*



Consider the proper knot in R^3 shown in Figure 4. One can use techniques developed by Churchard and Spring in reference [2] to isotope f to a p. l. proper knot; one merely places a p. l. 3-ball B around one of the tame points of the image of f in a manner such that $(B, B \cap \{Image(f)\})$ is a standard ball pair. One then "blows up" B and "combs the knotted part of the proper knot to infinity". On the other hand, the image of f does not run between the opposite ends of any properly embedded p. l. $D^2 \times R^1$. This can be seen as follows: if such a $D^2 \times R^1$ existed, there would be some positive integer k such that the subarcs A_k and A_{-k} would lie on the opposite sides of some properly embedded disk $D^2 \times \{t\}$. Hence, one could obtain disjoint 3-balls B and B' such that $A_k \subset B$ and $A_{-k} \subset B'$. However, it is shown in reference [4] (p. 166) that the arcs A_k and A_{-k} are unsplittable.

We now present a classification theorem for proper knots in solid open handlebodies with a deleted point.

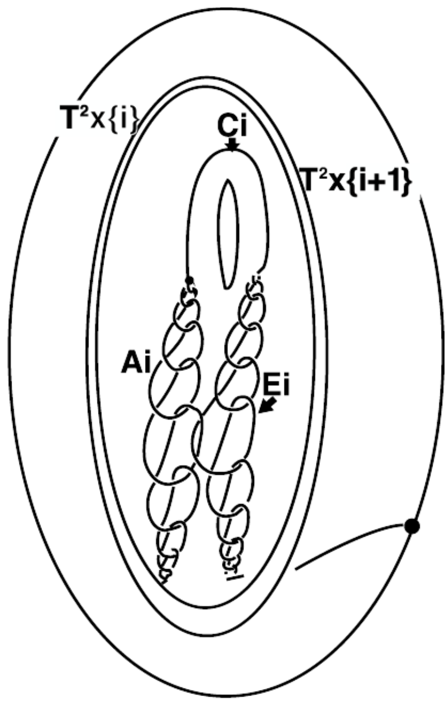
Theorem 3.7. *Let* M' *be an open solid handle body with at most a countable number of handles and let* $M = M' - B$ *(where B is a p. l. 3-ball in the interior of M). If the "boundary of the handlebody end" is* Γ_1 *and the "boundary of the deleted ball end" is* Γ_2, *then up to* *orientation and equivalence, there is a unique proper knot running between* Γ_1 *and* Γ_2.

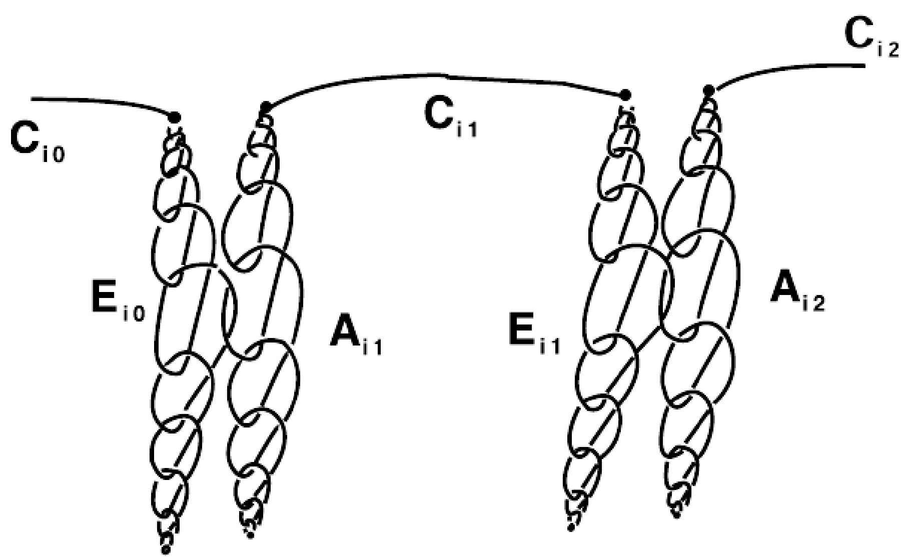
Proof. Let G be the one dimensional spine of M' and let $N(G)$ denote a regular neighborhood of G in M'. Without loss of generality, we may assume that the collar of the deleted 3-ball end E_1 is contained in $N(G)$. By Corollary 3.5, we may assume that f follows some collar ray of E_1 and, by general position, misses G. Hence, we can assume that the image of f contains a properly embedded collar ray $s \times (-\infty, a]$ $(s \in S^2, a < 0)$ which runs to Γ_2 and, for some $\epsilon > 0$, meets every product level $G \times (0, \epsilon)$ transversely at one point. Consequently, we can assume that $Image(f) \cap (M' - N(G))$ has at least one tame point.

It follows from the techniques of Proposition 2.2 of [3] that f can be isotoped to a p. l. proper knot which, outside of $G \times (0, \frac{\epsilon}{2})$, follows a collar ray of $E_1, w \times (\frac{\epsilon}{2}, \infty)$ $(w \in W_1)$; one uses the collar product structure to comb the proper knot to infinity at E_1. Hence the image of f can be properly isotoped to a proper knot g so that $Im\{g\} = \{w \times (\frac{\epsilon}{2}, \infty)) \cup \{s \times (-\infty, b]\}$ where $(s, b) \in E_2$ and $(w, \frac{\epsilon}{2}) \in E_1$ represent the same point in M. □

Example 3.8. *A proper knot can fail to lie on a properly embedded* $\mathsf{D}^2 \times \mathsf{R}^1$ *even if it runs between different ends of an open* 3*-manifold.*

Let $M = \mathsf{T}^2 \times \mathsf{R}$ where T is a torus. Think of M as being built up as $\cup_{\infty > i > -\infty} T_i$ where $T_i = \mathsf{T}^2 \times [i, i+1]$. Let T_i' denote the solid torus $T_i \cup \{\cup_{\infty > j > i} T_j\}$. Consider the proper knot f whose image is built up by the arcs $A_i \cup C_i \cup E_i$ as depicted in Figure 5. Subarcs A_i and E_i cannot be split from each other by a ball. Claim: the arc $R_i = A_i \cup C_i \cup E_i$ lies in no 3-ball $B \subset T_i'$. Proof of claim: suppose a ball B existed. Then look at the universal cover $\widehat{T}$ obtained by splitting T_i along a meridional disk which intersects both A_i and E_i in two points near where they "link". See Figure 6. B lifts to disjoint preimages ...$\widehat{B}_{-1}, \widehat{B}_0, \widehat{B}_1, \widehat{B}_2 \ldots$ each of which can contain only a finite number of components of $\widehat{R}_i$ which we denote by $\widehat{R}_{ij}$. Hence, in $\widehat{T}$, there exists an index k such that $\widehat{R}_{ik}$ is split from $\widehat{R}_{ik+1}$ by a ball. This is impossible. Therefore the arc R_i does not lie in any 3-ball in M. Hence it is impossible for the image of f to run between the opposite ends of any properly embedded $D^2 \times \mathsf{R}$.





We conclude this paper with a result that allows us to smooth certain types of proper knots that run to a sphere end. Recall that a proper knot f pierces a disk at p if there exists a p. l. disk D where ∂D links f (in the sense that ∂D cannot be shrunk to a point in the complement of the image of f) and $D \cap \{Im f\} = p$.

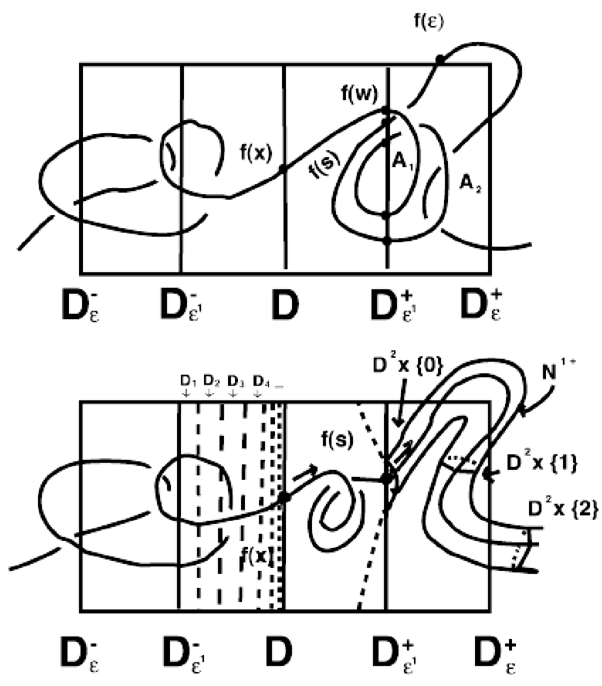
Proposition 3.9. *Let f be a proper knot in M^3 which pierces a disk at each of its points, runs to a sphere end Γ_2 and whose wild points are isolated. Then there is a proper isotopy* *that takes f to a proper knot g which pierces a disk at each of its points and follows a collar ray in* Γ_2.

Proof. Notice that Proposition 3.9 does not follow from Corollary 3.5 as the "collar ray following" proper knot promised by the corollary may well fail to pierce a disk at some of its points. We start our proof by proving the following:

Lemma 3.10. *Suppose f is a proper knot which runs from an end* Γ_1 *to a sphere end* Γ_2*, pierces a disk at each of its points and has n* $(0 < n < \infty)$ *wild points in* $\Sigma \times [0,\infty) \subset E_2$ *where* Σ *is some level two-sphere in* E_2*. Then f is properly isotopic to a proper knot* f' *which pierces a disk at each of its points and has* $n-1$ *wild points in* $\Sigma \times [0,\infty)$ *and follows a collar ray of* E_2*. Furthermore, if the first point of f to hit* $\Sigma \times \{0\}$ *is f(y),* f' *can be chosen so that* $f((-\infty,y]) = f'((-\infty,y])$.

Proof. First note that f is properly isotopic to a proper knot which follows a collar line of E_2. One merely notices that there is a level 2-sphere $\Sigma' = S^2 \times \{0\}$ of E_2 for which $f|\{f^{-1}(S^2 \times [0,\infty))\}$ contains no wild points. Therefore f can be assumed to be a p. l. embedding and Proposition 3.3 of [3] can be applied to "straighten out the end of f" in E_2.

Order the wild points of $f([y,\infty))$ and let $p = f(x)$ be the last wild point of f ($f(x,\infty)$ contains no wild points). Then there exists some x_1 and x_2 where $y < x_1 < x < x_2$ such that $f([x_1,\infty))$ has only one wild point p and $f([x_2,\infty))$ has no wild points. We will show how to construct a proper knot f_1 which is equivalent to f where $f_1((-\infty,x_1]) = f((-\infty,x_1])$ and $f_1([x_1,\infty))$ is p. l..



First, we will fix notation. See Figure 7. If D is a disk pierced by f at $p = f(x)$, D_β denotes the product $D \times [-\beta,\beta]$, D_β^+ denotes $D \times [0,\beta]$ and D_β^- denotes $D \times [-\beta,0]$. Assume that D_β^+ meets $f([x,x+\delta))$ and D_β^- meets $f((x-\delta,x])$ for all $\delta > 0$. Claim: There exists a $\sigma > 0$ such that $D_\sigma^+ \cap f((-\infty,x]) = \emptyset$ and $D_\sigma^-([x,\infty)) \cap f([x,\infty)) = \emptyset$. Proof of claim. Given $\sigma' > 0$, choose $\delta > 0$ so that $f([x-\delta,x+\delta]) \subset D_{\sigma'}$. Because D separates $D_{\sigma'}$ and D meets the image of f only at $f(x)$, $f([x-\delta,x)) \subset D_{\sigma'}^-$. Because f is proper, $f((-\infty,x-\delta]) \cap D_{\sigma'}$ is a compact set which misses D. Therefore there is some $\epsilon \leq \epsilon'$ such that

$f((-\infty, x-\delta]) \cap D^+_\sigma = \emptyset$. The same argument works for $D^-_\sigma([x,\infty))$ and $f([x,\infty))$.

Next choose $\epsilon > 0$ so that $D^+_\epsilon \cap f((-\infty,x]) = \emptyset$ and $D^-_\epsilon([x,\infty)) \cap f([x,\infty)) = \emptyset$ and let $t > x$ be the first point of $[x,\infty)$ where $f(t)$ meets $\partial(D\epsilon^+ - D)$. Assume that the intersection is transverse to $\partial(D\epsilon^+ - D)$. Choose $0 < \epsilon' < \epsilon$ so that $\partial D \times [-\epsilon', \epsilon']$ is disjoint from the image of f and that $f([t,\infty))$ is disjoint from $D^+_{\epsilon'}$. Let $w = min\{y \in \mathsf{R}, y \in f^{-1}(D \times \{\epsilon'\})$ and $s = max\{y \in \mathsf{R}, y \in f^{-1}(D^+_{\epsilon'})\}$. Note that $f^{-1}(D^+_{\epsilon'}) \subset [x,s] \subset [x,t]$ and $f(s) \in D \times \{\epsilon'\}$. Suppose that $w \neq s$. Since $f([w,s])$ is a tame arc which lies entirely in D^+_ϵ, we can assume that $f([w,s]) \cap (D \times [\epsilon', \epsilon])$ is a finite collection of p. l. arcs, say $A_1, \ldots A_k$, each of which has a tubular neighborhood. Because f eventually follows a collar ray to a sphere end Γ_2, it is possible to unlink each A_i with the image of f by repeatedly using the lamp cord trick (the "lassos of reference [3]). We isotope small subarcs of the A_i along a regular neighborhood of $f([s,\infty))$ to a level two-sphere which the image of f intersects only once. Then one can, by an ambient isotopy, take each A_i into the interior of $D^+_{\epsilon'}$ while leaving $f((-\infty,x]) \cup f([s,\infty))$ fixed. Now $f([x,s])$ is a properly embedded arc in $D^+_{\epsilon'}$. Therefore, we can adjust f by an ambient isotopy so that $w = s$.

Now we can assume that $Im(f) \cap D \times \{\epsilon'\} = s$. Next let N'^+ denote a tubular neighborhood of $f([s,\infty))$. Then $D_{\epsilon'}$ can be glued to N'^+ in a standard way as to form a properly embedded $\mathsf{D}^2 \times \mathsf{R}^+$ which contains $f([x,\infty))$ as a properly embedded ray with $f(x) \cap (\mathsf{D}^2 \times \{1\}) = p$ and $\mathsf{D}^2 \times \{0\}$ identified with $D \times \{-\epsilon'\}$ and $D^2 \times \{1\}$ identified with $D \times \{0\}$. Denote this $\mathsf{D}^2 \times \mathsf{R}^+$ by N^+.

Recall that ϵ' was chosen so that $Im(f((-\infty,x)) \cap \partial(D^-_{\epsilon'}) \subset D \times \{-\epsilon'\}$. Partition $D^-_{\epsilon'}$ by level disks $D \times \{-\frac{\epsilon'}{n}\}$ for $n \in \{1,2,3,\ldots\}$ and denote $D \times [\frac{-\epsilon'}{n}, \frac{-\epsilon'}{n+1}]$ by D_n. There is an ambient isotopy of M which is the identity outside of a regular neighborhood of N'^+ which takes D_1 to $\mathsf{D}^2 \times [0,1]$ and $\cup_{\infty > j \geq 2} D_j$ into $\mathsf{D}^2 \times [1,2)$. Call this ambient isotopy F_1. This is very similar to the isotopy described in the proof of Theorem 3.1. We can then follow F_1 by an ambient isotopy F_2 which is the identity on $\mathsf{D}^2 \times [0,1]$, takes $F_1(D_2)$ to $\mathsf{D}^2 \times [1,2]$ and $F_1(\cup_{\infty > j \geq 3} D_j)$ into $\mathsf{D}^2 \times [2,3)$. In a similar way, we define an ambient isotopy F_k which is the identity on $\mathsf{D}^2 \times [k-2, k-1]$, takes $F_{k-1}(D_k)$ to $\mathsf{D}^2 \times [k-1,k]$ and $F_{k-1}(\cup_{\infty > j \geq k+1} D_j)$ into $\mathsf{D}^2 \times [k, k+1)$. Concatenating the isotopies in a standard way and applying a standard reparametrization yields an isotopy that takes f to a proper knot f'. Note that if $x_1 \in \mathsf{R}$ is the first point of f to meet $D \times \{\epsilon'\}$ then $f((-\infty,x_1]) = f'((-\infty,x_1])$ and f' has one less wild point than f. The Lemma is proved. □

To prove the Proposition, we start by finding an unbounded set of level two spheres $\Sigma_i \in E_2$ arranged in such a way so that $max(f^{-1}(\Sigma_i)) < min(f^{-1}(\Sigma_{i+1}))$ for all $i \geq 1$. With no loss of generality, we can assume that the Σ_i miss the wild points of f. Let f_i denote the proper knot that agrees with f for all $x \leq max(f^{-1}(\Sigma_i))$ and follows a collar ray afterward. Note for all $j > i > 0$, f_j agrees with f_i for all $x \leq max(f^{-1}(\Sigma_i))$.

By applying the Wild Combing Lemma (Lemma 3.10) a finite number of times, we see that for all $i > 0$, there exists a proper isotopy $F^i : \mathsf{R} \times [0,1] \to M^3$ such that $F^i(y,0) = f_i(y)$, $F^i(y,1) = f_{i+1}(y)$ and for all $x \leq min(f^{-1}(\Sigma_i))$ and $t \in [0,1]$, $F^i(x,t) = f(x)$.

We now concatenate the proper isotopies F^i to obtain a proper isotopy F which takes the proper knot f_1 to f in the following way: let $\sigma_i : [\frac{i-1}{i}, \frac{i}{i+1}] \to [0,1]$ be defined by $\sigma_i(t) = i(i+1)(t - \frac{i-1}{i})$. Then define $F : \mathsf{R} \times [0,1] \to M$ by

$$F(x,t) = \begin{cases} F^i(x, \sigma_i(t)) \text{ for } t \in [\frac{i-1}{i}, \frac{i}{i+1}] (i \in \{1,2,3 \ldots\}) \\ f(x) \text{ for } t = 1 \end{cases}$$

. It is clear that F is continuous.

To see that F is proper we suppose that K is a compact subset of M and show that $wd(F^{-1}(K))$, defined as $wd(F^{-1}(K)) = max\{t | F(x,t) \cap K \neq \emptyset\} - min\{t | F(x,t) \cap K \neq \emptyset\}$ is bounded. First note that $wd(F^{-1}(K \cap closure\{M - \{\Sigma \times [0,\infty)\}\}) = wd(f^{-1}(K \cap closure\{M - \{K \times [0,\infty)\}\})$ because $F(x,t) = f(x)$ for all $((x,t)$ where $f(x) \notin \Sigma \times [0,\infty))$. So we need only check for $K \in \Sigma \times [0,\infty)$. Since K is compact, $K \subset \Sigma \times [0,m]$ for some $m \in \{1,2,\ldots\}$. Consequently, there exists an $i \in \{1, 2, 3, \ldots\}$ such that for all $j \geq i$, $(F^j)^{-1}(K) = f^{-1}(K)$. So, $wd(F^{-1}(K)) \leq (wd(f^{-1}(K) + wd((F^1)^{-1}(K)) + \ldots + wd((F^i)^{-1}(K)))$ and therefore is finite. Hence F is proper and the Proposition is proved.□

Theorem 3.11. *Suppose f is a proper knot which runs from a collared end E_1 to a sphere end E_2. Suppose f pierces a disk at all of its points and that its wild points are isolated. Then f is properly isotopic to a smooth (or p. l.) proper knot.*

Proof. From Proposition 3.9 it can be assumed that f follows a collar ray in E_2. We can also assume that all of the wild points of f lie in E_1. Because the wild points of f are isolated, f has only a finite number of wild points in $\Sigma \times [0,m] \subset E_1$ where Σ is a level surface (possibly of infinite genus). Let $\sigma \subset \mathsf{R}$ denote the smallest closed interval containing $f^{-1}(\Sigma \times [0,m])$. Then $f(\sigma)$ is an embedded arc with only a finite number of wild points. These can be ordered by the order of their preimages in σ. Call these points $x_1, x_2, \ldots, x_k$. First use Lemma 3.10 to comb the first wild point $f(x_1)$ to infinity in E_2; the details are exactly as in Lemma 3.10. Subsequently comb $x_2, x_3, \ldots\ldots x_k$. Hence, we can assume that f is equivalent to a proper knot f' where $f'(\sigma)$ is a smoothly embedded proper arc. We now use Proposition 3.5 of [3] to properly isotope f' to a proper knot which intersects a level surface $\Sigma' \times \{t\} \subset \Sigma \times [0,m]$ in exactly one point: basically one gets a convenient projection of f' and then repeatedly uses the "lamp cord" trick around a level two-sphere to change all crossings to undercrossings and then straightens f' so as to hit a selected level surface exactly once.

Next, one uses Σ' to comb all of the part of f' lying in $\Sigma' \times [t,\infty)$ to infinity in E_1 to obtain a proper knot f'' which follows a collar ray in E_1. f'' is equivalent to f and contains no wild points. □

The following corollaries are obtained from Theorem 3.11 and from Theorem 4.3 of reference [3].

Corollary 3.12. *Suppose f and g are proper knots which run from a collared end E_1 to a sphere end E_2. Suppose f and g pierce a disk at all of their points and their wild points are isolated. Then if f and g are connected by a proper homotopy, then they are connected by a proper isotopy. That is, in this case, proper homotopy implies proper isotopy.*

Proof. Let f' denote the smooth proper knot that is equivalent to f and g' the smooth proper knot that is equivalent to g. It is easy to see that f' is connected to g' by a proper homotopy H. The Smooth Approximation Theorem implies that there is a smooth proper homotopy which is homotopic to H that connects f' to g'. It follows from Theorem 4.3 of [3] that f' and g' are connected by a smooth proper isotopy. Hence, f can be connected to g by a proper isotopy. □

We now state a corollary concerning arbitrary topological proper knots which run from a collared end to a sphere end.

Corollary 3.13. *Suppose f and g are proper knots which run from a collared end E_1 to a sphere end E_2. If f is connected to g by a proper homotopy and f' is a smooth proper knot which is properly homotopic to f and g' is a smooth proper knot which is properly homotopic to g, then f' and g' are connected by a smooth, proper isotopy. In other words, if f and g are homotopic, then their smooth homotopy representatives are equivalent.*

4. Questions. Note that the classification of proper knots running between the opposite ends of $\mathsf{S}^2 \times \mathsf{R}$ is complete, both in the smooth and in the topological category. However, the question of whether all smooth proper knots which run between the opposite ends of a $\mathsf{D}^2 \times \mathsf{R}$ are equivalent in the smooth category (i. e., are connected by a smooth proper isotopy) remains open.

The topological proper knot classification for proper knots in manifolds other than $\mathsf{S}^2 \times \mathsf{R}$

or $D^2 \times R$ is, to the author's knowledge, unknown. In fact, the classification of smooth proper knots which run between the opposite ends of $F^2 \times R$ ($F^2 \neq S^2$) (a question posed by Churchard and Spring in [3]) remains open.

Are there any non-smoothable proper knots? Such a proper knot would have to either fail to pierce a disk at some of its points, or fail to be locally homogenous. The proper knot of Example 3.8 is a possible candidate.

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